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Globally stable direct fuzzy model reference adaptive control

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Abstract

In the paper a fuzzy adaptive control algorithm is presented. It belongs to the class of direct model reference adaptive techniques based on a fuzzy (Takagi–Sugeno) model of the plant. The plant to be controlled is assumed to be nonlinear and predominantly of the first order. Consequently, the resulting adaptive and control laws are very simple and thus interesting for use in practical applications. The system remains stable in the presence of unmodelled dynamics (disturbances, parasitic high-order dynamics and reconstruction errors are treated explicitly). The global stability of the overall system is proven in the paper, i.e. it is shown that all signals remain bounded while the tracking error and estimated parameters converge to some residual set that depends on the size of disturbance and high-order parasitic dynamics. The proposed algorithm is tested on a simulated three-tank system. Its performance is compared to the performance of a classical MRAC.

Keywords: Fuzzy system models; Fuzzy control; Takagi–Sugeno model; Model reference adaptive control

1. Introduction

The problem of control of nonlinear plants has received a great deal of attention in the past. The problem itself is fairly demanding, but when the model of the plant is unknown or poorly known, the solution becomes considerably more difficult. Nevertheless, several approaches exist to solve the problem. One possibility is to apply adaptive control. Classical adaptive control schemes (in this paper, adaptive control algorithms for LTI plants developed by the end of the 1970s are referred to as classical) do not produce good results, although adaptive parameters try to track the “true” local linear parameters of the current operating point. To overcome this problem, classical adaptive control was extended in the 1980s and 1990s to time-varying [24] and nonlinear plants [10]. Since

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we restricted our attention mainly to nonlinear plants that are more or less time-invariant, the former approaches were not as relevant even though they produce better results than classical adaptive control. The main drawback of adaptive control algorithms for nonlinear plants is that they demand fairly good knowledge of mathematics and are thus avoided by practicing engineers.

Many successful applications of fuzzy controllers [20,18] have shown their ability to control nonlinear plants. Despite their practical success, it seems that general control design techniques are still not available. One obvious solution is to introduce some sort of adaptation into the fuzzy controller. The first attempts at constructing a fuzzy adaptive controller can be traced back to [13], where so-called linguistic self-organising controllers were introduced. Many approaches were later presented where a fuzzy model of the plant was constructed on-line, followed by control parameters adjustment [11,16,17]. The main drawback of these schemes was that their stability was not treated rigorously. The universal approximation theorem [27] provided a theoretical background for new fuzzy direct and indirect adaptive controllers [26,19,22] whose stability was proven using the Lyapunov theory. Most of the early controllers demanded full state measurement, which is usually quite an unrealistic assumption. In [23] the problem was avoided by using a high-gain observer. Since the latter causes the robustness of the system to decrease, parameter projection is included in adaptive law to regain robustness. Another important issue in fuzzy adaptive control is how to overcome the difficulty of nonlinearly parameterised observers. In most of the current fuzzy adaptive systems, parameters appear linearly in the parameterised fuzzy approximators. Some work in this direction has already been done [5].

Adaptive control based on neural networks is very similar (in some cases even equivalent) to fuzzy adaptive control. Research in this area has been very active in recent years. In [4,8,19], model reference adaptive neural-network-based controllers are presented and their stability is proven using the Lyapunov theory. In [3], two controllers are used: a linear robust adaptive one and a neural-network-based adaptive one. A switching mechanism is proposed to improve performance of the system, while robustness is still guaranteed.

In this paper, robustness analysis of the controlled system plays a central role. It is well known that a fuzzy system can approximate any continuous function but, in general, there is always a reconstruction error that acts as a disturbance in the adaptive law. The problem of disturbances and unmodelled dynamics is very well known in the adaptive community [15]. Robust adaptive control was proposed to overcome this [6,7]. Similar solutions have also been used in fuzzy adaptive controllers, i.e. projection [23], dead zone [9], etc. have been included in the adaptive law to prevent instability due to reconstruction error. In this paper, not only reconstruction error and disturbances but also error due to high-order parasitic dynamics (which are inevitable) is treated explicitly. The latter is especially problematic since it can become unbounded [7]. The rationale behind the study of the influence of parasitics is that the control plant is assumed to be nonlinear and predominantly of the first order (higher-order parasitics are catered for by the robustness properties of the controller). In our opinion, such plants occur quite often in process industries. Our assumption results in very simple control and adaptive laws. The interesting thing is that the proposed direct fuzzy model reference adaptive control (DFMRAC) algorithm greatly resembles the classical MRAC of the first-order plant. In fact, it can be obtained by fuzzification of control gains and the inclusion of e_1-modification [12] into the adaptive law. The stability of DFMRAC is examined thoroughly within the framework proposed by Ioannou and Sun [7]. The boundedness of estimated parameters, the tracking error and all the signals in the system is proven, as well as the convergence of the tracking error and estimated parameters to some residual set that depends on the size of disturbance and parasitic dynamics.

The paper is organised as follows. In Section 2, the class of plants that will be discussed in the paper is presented. In Section 3, a description of the proposed algorithm is given. The performance of the algorithm is tested on a simulated three-tank system in Section 4. The conclusions are presented in Section 5. In the appendices the proofs of the important theorems are given, together with the necessary background.

2. Plant model development

There are many approaches to nonlinear system identification in the literature. Among them, identification by means of fuzzy models is quite common. Since our aim was to use simple algorithms, the Takagi–Sugeno model was chosen to describe plant behaviour [21]. If the first order plant is assumed and the nonlinearity of the plant depends on two measurable quantities z_1 and z_2, the model is described by k *if–then* rules in the following form

$$\begin{aligned}&\textbf{if } z_1 \textbf{ is } \mathbf{A}_{i_a} \textbf{ and } z_2 \textbf{ is } \mathbf{B}_{i_b} \quad \textbf{then } \dot{y}_p = -a_i y_p + b_i u\\ &i_a = 1,\ldots,n_a;\ \ i_b = 1,\ldots,n_b;\ \ i = 1,\ldots,k,\end{aligned} \tag{1}$$

where u and y_p are the input and output of the plant, respectively, $\mathbf{A}_{i_a}$, $\mathbf{B}_{i_b}$ are fuzzy membership functions, and a_i and b_i are the plant parameters in the ith fuzzy domain. The antecedent variables that define the fuzzy domain in which the system is currently situated are denoted by z_1 and z_2 (in actual fact there can be only one such variable and there can also be more, but this does not affect the approach described in this paper). There are n_a and n_b membership functions for the first and the second antecedent variables, respectively. The product $k = n_a \times n_b$ defines the number of fuzzy rules. The membership functions have to cover the whole operating area of the system. The output of the Takagi–Sugeno model is then given by the following equation:

$$\dot{y}_p = \frac{\sum_{i=1}^{k}(\beta_i^0(\boldsymbol{\varphi})(-a_i y_p + b_i u))}{\sum_{i=1}^{k}\beta_i^0(\boldsymbol{\varphi})}, \tag{2}$$

where $\boldsymbol{\varphi}$ represents the vector of antecedent variables z_i. The degree of fulfilment $\beta_i^0(\boldsymbol{\varphi})$ is obtained using the T-norm, which in this case is a simple algebraic product of membership functions

$$\beta_i^0(\boldsymbol{\varphi}) = T(\mu_{A_{i_a}}(z_1), \mu_{B_{i_b}}(z_2)) = \mu_{A_{i_a}}(z_1)\cdot\mu_{B_{i_b}}(z_2), \tag{3}$$

where $\mu_{A_{i_a}}(z_1)$ and $\mu_{B_{i_b}}(z_2)$ stand for degrees of fulfilment of the corresponding membership functions. The degrees of fulfilment for the whole set of rules can be written in compact form

$$\boldsymbol{\beta}^0 = [\beta_1^0 \quad \beta_2^0 \quad \ldots \quad \beta_k^0]^{\mathrm{T}} \tag{4}$$

and given in normalised form as

$$\boldsymbol{\beta} = \frac{\boldsymbol{\beta}^0}{\sum_{i=1}^{k}\beta_i^0}. \tag{5}$$

Due to (2) and (5), the first-order plant can be modelled in fuzzy form as

$$\dot{y}_p = -(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{a})y_p + (\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b})u, \tag{6}$$

where $\mathbf{a} = [a_1\ a_2\ \cdots\ a_k]^{\mathrm{T}}$ and $\mathbf{b} = [b_1\ b_2\ \cdots\ b_k]^{\mathrm{T}}$ are vectors of unknown plant parameters in respective fuzzy domains.

To assume that the controlled system is of the first order is a quite huge idealisation; parasitic dynamics are therefore included in the model of the plant. The linear time-invariant system of the first order with stable factor plant perturbations is described by the following equation:

$$y_p(s) = \frac{b/(s+c) + \Delta_1(s)}{(s+a)/(s+c) + \Delta_2(s)}\, u(s), \tag{7}$$

where $b/(s+a)$ is the transfer function of the nominal system, c is a positive constant, $\Delta_1(s)$ and $\Delta_2(s)$ are stable transfer functions [25], and $u(s)$ and $y_p(s)$ are the Laplace transforms of the plant's input and output, respectively. By multiplying the numerator and denominator of (7) by $(s+c)$, the following is obtained:

$$y_p(s) = \frac{b + \Delta_u'(s)}{s + a + \Delta_y'(s)}\, u(s), \tag{8}$$

where the definition of $\Delta_u'(s)$ and $\Delta_y'(s)$ follows directly. Since a and b in (8) are not known, they can be found such that $\Delta_u'(s)$ and $\Delta_y'(s)$ are definitely strictly proper transfer functions.[1] Eq. (8) can be rewritten as

$$s y_p = -a y_p + bu - \Delta_y'(s) y_p + \Delta_u'(s) u. \tag{9}$$

By taking into account the fuzzy model of plant (6), the first two terms in (9) that apply to linear systems are replaced and the plant model becomes:

$$\dot{y}_p = -(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{a})y_p + (\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b})u - \Delta_y'(p) y_p + \Delta_u'(p) u, \tag{10}$$

where p is a differential operator $\mathrm{d}/\mathrm{d}t$, while $\Delta_y'(p)$ and $\Delta_u'(p)$ are linear operators in the time domain that are equivalent to transfer functions $\Delta_y'(s)$ and $\Delta_u'(s)$. It is assumed that the plant is also disturbed by an external disturbance and the final model of the plant used in this paper is obtained by adding the disturbance d' to (10):

$$\dot{y}_p = -(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{a})y_p + (\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b})u - \Delta_y'(p) y_p + \Delta_u'(p) u + d'. \tag{11}$$

Assumptions on the plant model (11):

(A1) Absolute values of the elements of vector $\mathbf{b}$ are bounded from below and from above: $b_{\min} < |b_i| < b_{\max}$, $i = 1, 2, \ldots, k$ and $b_{\min}$ and $b_{\max}$ are some positive constants.

(A2) Absolute values of the elements of vector $\mathbf{a}$ are bounded from above: $|a_i| < a_{\max}$, $i = 1, 2, \ldots, k$ and $a_{\max}$ is a positive constant.

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[1] If they are only biproper, a solution with different a, b, $\Delta_u'(s)$ and $\Delta_y'(s)$ can always be found such that $\Delta_u'(s)$ and $\Delta_y'(s)$ are strictly proper and (8) still holds.

(A3) The signs of the elements in vector **b** are the same.

Some comments on the above assumptions:

If some b_i approached 0, the system would become almost uncontrollable in that operating point. We know that uncontrollability is not easily circumvented in any type of control, especially not in adaptive control; the first part of **A1** ('bounded from below' part) therefore has to hold. The same goes for the consequences of violation of assumption **A3**, namely some operating points (characterised by $\boldsymbol{\beta}$) would exist where the gain of the linearised plant $\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b}$ was 0 if the elements in **b** were not of the same sign. Moreover, the gain of the linearised plant would be positive in some operating points and negative in others. The control of such a plant would always be a problem and our attention is not directed to plants of that kind. Although fuzzy models can be regarded as universal approximators, only arbitrary small modelling errors are attainable in general. That is why over-large elements of **a** or **b** would cause large modelling errors (the second part of **A1**—the 'bounded from above' part—and **A2** have to hold).

It is worth mentioning that only dominant plant dynamics are assumed to be nonlinear while parasitic dynamics are linear. This is not a too unrealistic assumption since only the upper bound on the certain norms of the unmodelled dynamics are used in the theorem given later on in the paper. If the nonlinearity of the unmodelled dynamics is not too obvious, the proposed plant model is sufficient and can be used in quite a broad range of real plants, especially in process industries where first-order nonlinear systems are quite common.

The prerequisite for using model (11) is that we know what system variables the nonlinearity depends upon, i.e. what signals (z_1 and z_2 in this section) influence the calculation of $\boldsymbol{\beta}$. The choice of these so-called fuzzification or antecedent variables depends on the plant behaviour and is a similar problem to that of structural identification [21] in the case of the Takagi–Sugeno model. In [21] it was proposed that these variables were system input and system output. Since the realisation of control is not possible if $\boldsymbol{\beta}$ depends on u, $\boldsymbol{\beta}$ has to be calculated by means of y_p and/or some other signal(s) that might be correlated with the change in the system dynamics. Since the choice of fuzzification variables does not influence the form of the model (11) and the algorithm proposed later, it will not be addressed in the paper.

3. Proposed direct fuzzy model reference adaptive control algorithm

In the previous section the model of the plant was described. The first two terms on the right-hand side of (11) will serve as a model for control design, while the other terms will be catered for by the robustness properties of the adaptive and control laws since they are unknown in advance. It has to be pointed out that **a** and **b** are also unknown. To overcome this difficulty, adaptive control will be used.

The question still remains whether to use a direct or indirect adaptive scheme. Both approaches have some advantages and some disadvantages that are well known and documented for the adaptive control of LTI plants (e.g. [7]). Since it is our belief that it is much harder to prove global stability in the latter case, direct adaptive control was used in our approach, i.e. control parameters were estimated directly by using measurable signals. The task of this section is to find the control and adaptive laws that suit the design objective.

It was mentioned that the proposed approach to fuzzy adaptive control greatly resembles the classical MRAC. Since our attention is focused on plants that are predominantly of the first order, the MRAC of the first-order LTI plant will be recalled first. The control algorithm will later be extended to nonlinear plants of the first order with high-order parasitics.

3.1. MRAC of LTI plants

Let us briefly recall the classical approach to MRAC of the first-order linear time-invariant system. The approach described below is based on the Lyapunov theory and can be found in most of the textbooks on adaptive control (e.g. [2]).

The LTI plant of the first order can be described by means of the differential equation

$$\dot{y}_p = -a y_p + bu, \tag{12}$$

where u and y_p are the input and the output of the plant, respectively, while a and b are unknown constants. By choosing reference model

$$\dot{y}_m = -a_m y_m + b_m w \tag{13}$$

a control law

$$u = fw - q y_p \tag{14}$$

follows to achieve the design objective, where w is the reference signal. The classical solution to find the correct values for control parameters f and q is to estimate them by means of the following adaptive law:

$$\dot{f} = -\gamma_f \operatorname{sgn}(b) e w,$$
$$\dot{q} = \gamma_q \operatorname{sgn}(b) e y_p, \tag{15}$$

where e is the tracking error, defined as the difference between y_p and y_m, while γ_f and γ_q are arbitrary positive constants, usually referred to as adaptive gains.

As shown by Rohrs et al. [15], the above approach is not robust with respect to high-order unmodelled dynamics and disturbances; therefore the adaptive law or the control law or external excitation has to be changed to achieve robustness. As will be shown later on in the paper, our approach was to use modified adaptive law.

3.2. DFMRAC for the class of nonlinear plants

The reason for presenting MRAC for the first-order linear plant in the previous section is that the proposed DFMRAC algorithm is a straightforward extension of the former. The latter assumes the fuzzification of the forward gain f and the feedback gain q. The fuzzified gains are described by means of fuzzy numbers $\mathbf{f}$ and $\mathbf{q}$

$$\mathbf{f}^{\mathrm{T}} = [f_1 \quad f_2 \quad \cdots \quad f_k],$$
$$\mathbf{q}^{\mathrm{T}} = [q_1 \quad q_2 \quad \cdots \quad q_k], \tag{16}$$

where k stands for the number of fuzzy rules, as mentioned before. The reference model is the same as in (13)

$$\dot{y}_m = -a_m y_m + b_m w. \tag{17}$$

The control law is obtained by slightly extending (14); namely, scalar control gains are substituted by vector ones:

$$u = (\boldsymbol{\beta}^{\mathrm{T}}\mathbf{f})w - (\boldsymbol{\beta}^{\mathrm{T}}\mathbf{q})y_p. \tag{18}$$

The tracking error is the same as before

$$e = y_p - y_m. \tag{19}$$

3.2.1. Adaptive law

The most important part of the algorithm is the adaptive law that can be put down in scalar form

$$\dot{f}_i = -\gamma_{fi} b_{\mathrm{sign}}\, \varepsilon w \beta_i - \gamma_{fi}|\varepsilon m| v_0 f_i \beta_i, \quad i = 1,2,\ldots,k,$$
$$\dot{q}_i = \gamma_{qi} b_{\mathrm{sign}}\, \varepsilon y_p \beta_i - \gamma_{qi}|\varepsilon m| v_0 q_i \beta_i, \quad i = 1,2,\ldots,k \tag{20}$$

or in equivalent vector form, which is more suitable for analysis due to its compactness

$$\dot{\mathbf{f}} = -\boldsymbol{\Gamma}_f b_{\mathrm{sign}} \varepsilon w \boldsymbol{\beta} - \boldsymbol{\Gamma}_f |\varepsilon m| v_0 \mathbf{F}\boldsymbol{\beta},$$
$$\dot{\mathbf{q}} = \boldsymbol{\Gamma}_q b_{\mathrm{sign}} \varepsilon y_p \boldsymbol{\beta} - \boldsymbol{\Gamma}_q |\varepsilon m| v_0 \mathbf{Q}\boldsymbol{\beta}, \tag{21}$$

where γ_{fi} and γ_{qi} are positive scalar adaptive gains, ε is the error that will be defined later, m is a variable for normalisation to be defined, v_0 is a design parameter that determines the influence of the 'leakage' [7], $\mathbf{F} = \mathrm{diag}(\mathbf{f})$, $\mathbf{Q} = \mathrm{diag}(\mathbf{q})$, and $\boldsymbol{\Gamma}_f$ and $\boldsymbol{\Gamma}_q$ are diagonal matrices of the corresponding adaptive gains γ_{fi} and γ_{qi}, respectively. If the sign of the elements in vector $\mathbf{b}$ in (11) is negative, b_{sign} is -1; otherwise it is $+1$. By introducing $\boldsymbol{\theta}^{\mathrm{T}} \triangleq [\mathbf{f}^{\mathrm{T}}\ \mathbf{q}^{\mathrm{T}}]$ and $\boldsymbol{\omega}^{\mathrm{T}} \triangleq [\boldsymbol{\beta}^{\mathrm{T}} w\ -\boldsymbol{\beta}^{\mathrm{T}} y_p]$, (21) can be made even more compact

$$\dot{\boldsymbol{\theta}} = -\boldsymbol{\Gamma} b_{\mathrm{sign}} \varepsilon \boldsymbol{\omega} - \boldsymbol{\Gamma}|\varepsilon m| v_0 \boldsymbol{\theta}_d \boldsymbol{\beta}, \tag{22}$$

where $\boldsymbol{\Gamma}$ is the diagonal matrix of scalar adaptive gains and $\boldsymbol{\theta}_d^{\mathrm{T}} = [\mathbf{F}^{\mathrm{T}}\ \mathbf{Q}^{\mathrm{T}}]$.

There are a number of remarks concerning the adaptive law (22) that must be made here. The first term on the right-hand side of (22) is equivalent to the adaptive law (15). The second term introduces leakage, more specifically so-called e_1-modification [12]. Note that instead of the product $\boldsymbol{\theta}_d\boldsymbol{\beta}$, only $\boldsymbol{\theta}$ is used in [12], where the situation was simpler since the plant was LTI. The difference is seen more clearly from (20). When the system leaves a certain operating region (fuzzy domain), the corresponding membership function β_i becomes 0. If β_i was not included in the second term on the right-hand side of (20), the system would gradually forget the estimated parameter values f_i and q_i. When the system returned to the operating region, it would use the wrong parameter estimates. By including β_i in the second term on the right-hand side of (20), the adaptation of the respective parameter freezes until β_i is non-zero. This difference makes the analysis of the properties of adaptive law a little different than that performed by Ioannou and Sun [7]. On the other hand, the classical demand on the excitation of the external signal that prevents parameter drift is relaxed a little since some parameters are frozen at each instant and only those that correspond to the current fuzzy domains are potential candidates for the undesired adaptation (parameter drift).

Note that the adaptation is not governed by the tracking error e in (22). Instead, signal ε is used, which is defined as

$$\varepsilon = e - G_m(p)(\varepsilon n_s^2), \tag{23}$$

where $n_s^2 = m^2 - 1$ and $G_m(p)$ is the reference model operator in the time domain.

Theorem 1. *Adaptive law described by* (20) (*or equivalently* (21) *or* (22)) *guarantees the boundedness of the estimated parameter vectors* $\mathbf{f}$ *and* $\mathbf{q}$, *provided* m *is designed such that*

$$\frac{w}{m}, \frac{y_p}{m} \in \mathscr{L}_\infty. \tag{24}$$

Proof. Lyapunov-like function is chosen

$$V_{fi} = \frac{1}{2\gamma_{fi}} f_i^2. \tag{25}$$

Its derivative is

$$\begin{aligned} \dot{V}_{fi} &= \frac{1}{\gamma_{fi}} f_i \dot{f}_i = -b_{\text{sign}} \varepsilon w f_i \beta_i - |\varepsilon m| v_0 f_i^2 \beta_i \\ &= -|\varepsilon m| v_0 \beta_i \left(f_i^2 + b_{\text{sign}} \operatorname{sgn}(\varepsilon m) \frac{w}{m} \frac{1}{v_0} f_i \right). \end{aligned} \tag{26}$$

The derivative of the Lyapunov-like function (25) is non-positive if

$$|f_i| > \left| b_{\text{sign}} \operatorname{sgn}(\varepsilon m) \frac{w}{m} \frac{1}{v_0} \right| = \frac{1}{v_0} \left| \frac{w}{m} \right|. \tag{27}$$

Since $w/m \in \mathscr{L}_\infty$, $|f_i|$ is also bounded from above (it decreases until it reaches $(1/v_0)|w(t)/m(t)|$). In a similar manner it can be shown that q_i is bounded if $y_p/m \in \mathscr{L}_\infty$. Since the design of m is at the discretion of the designer, it can be concluded that estimated parameters are bounded, i.e. $\mathbf{f}, \mathbf{q} \in \mathscr{L}_\infty$. □

3.2.2. Error model

By subtracting (17) from (11), the following equation is obtained:

$$\begin{aligned} \dot{e} = &-a_m e + [(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b})(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{f}) - b_m]w - [(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b})(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{q}) + (\boldsymbol{\beta}^{\mathrm{T}}\mathbf{a}) - a_m]y_p \\ &+ \Delta_u'(p)u - \Delta_y'(p)y_p + d'. \end{aligned} \tag{28}$$

It is impossible to find such $\mathbf{f}$ and $\mathbf{q}$ that would make the expressions in square brackets equal to zero for a general case. This means that the perfect tracking of the reference model is not possible by any choice of the control vectors, even when no parasitic dynamics or disturbances are present. A decision has to be made as to what values for the elements of the vectors $\mathbf{f}$ and $\mathbf{q}$ are the desired ones. Those elements will be denoted by f_i^* and q_i^*. They will be obtained by making the expressions in the square brackets in (28) equal to zero:

$$(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b})(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{f}) - b_m = 0,$$
$$(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b})(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{q}) + (\boldsymbol{\beta}^{\mathrm{T}}\mathbf{a}) - a_m = 0. \tag{29}$$

As established before, a general solution for $\mathbf{f}$ and $\mathbf{q}$ in (29) does not exist. A particular solution will be found for cases where only one fuzzy domain is activated. This is done for all k fuzzy domains to obtain all f_i^*'s and q_i^*'s. Mathematically, this is done by setting $\boldsymbol{\beta} = [0 \ \cdots \ 0 \ 1 \ 0 \ \cdots \ 0]$ in (29), i.e. by choosing ith element of the vector $\boldsymbol{\beta}$ equal to 1 while others are equal to 0

$$b_i f_i^* - b_m = 0, \quad i = 1, 2, \ldots, k,$$
$$b_i q_i^* + a_i - a_m = 0, \quad i = 1, 2, \ldots, k. \tag{30}$$

This actually means that the desired control parameters are the same as they would be if obtained in each fuzzy domain separately. This also leads to perfect tracking if the plant is currently in only one fuzzy domain (local linear model of that domain applies) and there are no parasitic dynamics or disturbances. If some of the above conditions are violated, some terms on the right-hand side of (28) are non-zero. It will be shown that these terms do not affect the stability of the system.

Desired control parameters

$$\mathbf{f}^{*\mathrm{T}} = [f_1^* \quad f_2^* \quad \cdots \quad f_k^*],$$
$$\mathbf{q}^{*\mathrm{T}} = [q_1^* \quad q_2^* \quad \cdots \quad q_k^*] \tag{31}$$

are bounded due to (30) and the assumptions **A1** and **A2**. The parameter errors are defined as

$$\tilde{\mathbf{f}} = \mathbf{f} - \mathbf{f}^*,$$
$$\tilde{\mathbf{q}} = \mathbf{q} - \mathbf{q}^*. \tag{32}$$

Our wish is to change the expressions in the square brackets of (28) with new ones

$$(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b})(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{f}) - b_m = b\tilde{\mathbf{f}}^{\mathrm{T}}\boldsymbol{\beta} + b_m \frac{\eta_w}{w},$$
$$(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b})(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{q}) + (\boldsymbol{\beta}^{\mathrm{T}}\mathbf{a}) - a_m = b\tilde{\mathbf{q}}^{\mathrm{T}}\boldsymbol{\beta} + b_m \frac{\eta_y}{y_p}, \tag{33}$$

where

$$b = \inf_{\boldsymbol{\beta}} \boldsymbol{\beta}^{\mathrm{T}}\mathbf{b} = \min_i b_i. \tag{34}$$

By using (32) the first equation in (33) yields:

$$b\tilde{\mathbf{f}}^{\mathrm{T}}\boldsymbol{\beta} + b_m \frac{\eta_w}{w} = \boldsymbol{\beta}^{\mathrm{T}}\mathbf{b}\mathbf{f}^{\mathrm{T}}\boldsymbol{\beta} - b_m = \boldsymbol{\beta}^{\mathrm{T}}\mathbf{b}\mathbf{f}^{*\mathrm{T}}\boldsymbol{\beta} + \boldsymbol{\beta}^{\mathrm{T}}\mathbf{b}\tilde{\mathbf{f}}^{\mathrm{T}}\boldsymbol{\beta} - b_m. \tag{35}$$

Define matrix **B**:

$$\mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{bmatrix} [b_1^{-1} \quad b_2^{-1} \quad \cdots \quad b_k^{-1}] = \begin{bmatrix} 1 & \frac{b_1}{b_2} & \cdots & \frac{b_1}{b_k} \\ \frac{b_2}{b_1} & 1 & \cdots & \frac{b_2}{b_k} \\ \vdots & \vdots & \ddots & \\ \frac{b_k}{b_1} & \frac{b_k}{b_2} & & 1 \end{bmatrix}. \tag{36}$$

Using $[1\ 1\ \cdots\ 1]\boldsymbol{\beta}=1$ (see Eq. (5)) and Eq. (36), Eq. (35) yields:

$$\begin{aligned} b\tilde{\mathbf{f}}^{\mathrm{T}}\boldsymbol{\beta} + b_m\frac{\eta_w}{w} &= \boldsymbol{\beta}^{\mathrm{T}}\mathbf{b}\tilde{\mathbf{f}}^{\mathrm{T}}\boldsymbol{\beta} + \boldsymbol{\beta}^{\mathrm{T}}b_m\mathbf{B}\boldsymbol{\beta} - b_m[1 \quad 1 \quad \cdots \quad 1]\boldsymbol{\beta} \\ &= \boldsymbol{\beta}^{\mathrm{T}}\mathbf{b}\tilde{\mathbf{f}}^{\mathrm{T}}\boldsymbol{\beta} + b_m\{\boldsymbol{\beta}^{\mathrm{T}}\mathbf{B} - [1 \quad 1 \quad \cdots \quad 1]\}\boldsymbol{\beta}. \end{aligned} \tag{37}$$

The expression in the curly brackets is denoted by $\boldsymbol{\xi}^{\mathrm{T}}$. Since $0 \leqslant \beta_i \leqslant 1$, it follows:

$$\begin{aligned} &\min_j \frac{b_j}{b_i} - 1 \leqslant \xi_i \leqslant \max_j \frac{b_j}{b_i} - 1 \\ &\min_j \frac{b_j - b_i}{b_i} \leqslant \xi_i \leqslant \max_j \frac{b_j - b_i}{b_i} \\ &|\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{\beta}| \leqslant \frac{\max_{i,j} |b_j - b_i|}{\min_i |b_i|} < C_1 \end{aligned} \tag{38}$$

due to assumption **A1**, where C_1 is a constant. Error η_w can be deduced from (37)

$$\eta_w(t) = \frac{\boldsymbol{\beta}^{\mathrm{T}}(t)\mathbf{b} - b}{b_m}\tilde{\mathbf{f}}^{\mathrm{T}}(t)\boldsymbol{\beta}(t)w(t) + \boldsymbol{\xi}^{\mathrm{T}}(t)\boldsymbol{\beta}(t)w(t) = f_w(t)w(t), \tag{39}$$

where $f_w(t)$ was introduced. Since $\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b}$ (gain of the plant), $\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{\beta}$ and $\tilde{\mathbf{f}}$ are bounded (see assumption **A1** and Theorem 1), $|f_w|$ is always bounded, and it follows:

$$|\eta_w(t)| \leqslant |w(t)| \sup_t |f_w| = \bar{f}_w|w(t)|. \tag{40}$$

If the gain of the controlled plant does not depend very much on the antecedent variables (elements of the vector **b** are similar), $(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b} - b)$ and $\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{\beta}$ tend to zero and, consecutively, do $\bar{f}_w$ and η_w.

It follows from the second equation in (33)

$$\eta_y = \left[\frac{(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b})(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{q})}{b_m} + \frac{(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{a})}{b_m} - \frac{a_m}{b_m} - \frac{b\tilde{\mathbf{q}}^{\mathrm{T}}\boldsymbol{\beta}}{b_m}\right] y_p. \tag{41}$$

It will be shown next that the function in the square brackets in (41) is bounded. Let us take a look at the first term in the square brackets of (41)

$$\frac{(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b})(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{q})}{b_m} = \frac{\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b}\mathbf{q}^{*\mathrm{T}}\boldsymbol{\beta}}{b_m} + \frac{\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b}\tilde{\mathbf{q}}^{\mathrm{T}}\boldsymbol{\beta}}{b_m}$$

$$= \boldsymbol{\beta}^{\mathrm{T}} \left\{ \frac{a_m}{b_m}\mathbf{B} - \frac{1}{b_m} \begin{bmatrix} a_1 & a_2\frac{b_1}{b_2} & \cdots & a_k\frac{b_1}{b_k} \\ a_1\frac{b_2}{b_1} & a_2 & \cdots & a_k\frac{b_2}{b_k} \\ \vdots & \vdots & \ddots & \\ a_1\frac{b_k}{b_1} & a_2\frac{b_k}{b_2} & & a_k \end{bmatrix} \right\} \boldsymbol{\beta} + \frac{\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b}\tilde{\mathbf{q}}^{\mathrm{T}}\boldsymbol{\beta}}{b_m}. \tag{42}$$

The matrix in the square brackets will be denoted by $\mathbf{A}$ in the following. Eq. (41) can be rewritten as

$$\eta_y = \left[\boldsymbol{\beta}^{\mathrm{T}}\left(\frac{a_m}{b_m}\mathbf{B} - \frac{1}{b_m}\mathbf{A}\right)\boldsymbol{\beta} + \frac{\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b}\tilde{\mathbf{q}}^{\mathrm{T}}\boldsymbol{\beta}}{b_m} + \frac{(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{a})}{b_m} - \frac{a_m}{b_m} - \frac{b\tilde{\mathbf{q}}^{\mathrm{T}}\boldsymbol{\beta}}{b_m}\right] y_p$$

$$= \frac{a_m}{b_m}(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{B} - [1 \quad 1 \quad \cdots \quad 1])\boldsymbol{\beta} y_p - \frac{1}{b_m}\{\boldsymbol{\beta}^{\mathrm{T}}\mathbf{A} - \mathbf{a}^{\mathrm{T}}\}\boldsymbol{\beta} y_p + \frac{\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b} - b}{b_m}\tilde{\mathbf{q}}^{\mathrm{T}}\boldsymbol{\beta} y_p. \tag{43}$$

The expression in the parentheses in the first term is equal to $\boldsymbol{\xi}^{\mathrm{T}}$, while the expression in the curly brackets is denoted by $\boldsymbol{\chi}^{\mathrm{T}}$. Since $0 \leqslant \beta_i \leqslant 1$, the ith element of the vector $\boldsymbol{\chi}$ can be bounded from above and below:

$$\min_j a_i \frac{b_j - b_i}{b_i} \leqslant \chi_i \leqslant \max_j a_i \frac{b_j - b_i}{b_i}$$

$$|\boldsymbol{\chi}^{\mathrm{T}}\boldsymbol{\beta}| \leqslant \max_i a_i \frac{\max_{i,j} |b_i - b_j|}{\min_i b_i} < C_2 \tag{44}$$

due to assumptions **A1** and **A2**, where C_2 is a constant. Finally η_y can be expressed as

$$\eta_y = \frac{a_m}{b_m}\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{\beta} y_p - \frac{1}{b_m}\boldsymbol{\chi}^{\mathrm{T}}\boldsymbol{\beta} y_p + \frac{\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b} - b}{b_m}\tilde{\mathbf{q}}^{\mathrm{T}}\boldsymbol{\beta} y_p = f_y y_p. \tag{45}$$

It can be seen from (45) that the introduced f_y is a function of time. For our purposes, only the upper bound on $|f_y(t)|$ is important. Since $\boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{\beta}$, $\boldsymbol{\chi}^{\mathrm{T}}\boldsymbol{\beta}$, $\tilde{\mathbf{q}}$ (see Theorem 1) and $(\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b} - b)/b_m$ are bounded, it follows

$$|\eta_y(t)| \leqslant |y_p(t)| \sup_t |f_y(t)| = \bar{f}_y |y_p(t)|. \tag{46}$$

According to (45), $\bar{f}_y$ illustrates the nonlinearity of the plant (or better, its gain). If the elements of the vector $\mathbf{b}$ tend to a constant, $\bar{f}_y$ tends to 0. For linear plants or such that the gain of the plant is independent of $\boldsymbol{\beta}$, $\bar{f}_y$ is zero.

The final result follows from (28) and (33)

$$\dot{e} = -a_m e + b_m \left(\frac{b}{b_m} \tilde{\mathbf{f}}^{\mathrm{T}} \boldsymbol{\beta} w - \frac{b}{b_m} \tilde{\mathbf{q}}^{\mathrm{T}} \boldsymbol{\beta} y_p + \eta_w - \eta_y \right) + \Delta'_u(p)u - \Delta'_y(p) y_p + d'$$

$$= -a_m e + b_m \left(\frac{b}{b_m} \tilde{\mathbf{f}}^{\mathrm{T}} \boldsymbol{\beta} w - \frac{b}{b_m} \tilde{\mathbf{q}}^{\mathrm{T}} \boldsymbol{\beta} y_p + f_w w - f_y y_p + \Delta_u(p)u - \Delta_y(p) y_p + d \right), \quad (47)$$

where the introduction of $\Delta_u(p)$ and $\Delta_y(p)$ is obvious. In the following, the expression in parentheses in (47) shall be replaced by η to simplify the notation, i.e.

$$\eta(t) = f_w(t) w(t) - f_y(t) y_p(t) + \Delta_u(p) u(t) - \Delta_y(p) y_p(t) + d(t). \quad (48)$$

The expression in (47) is the so-called error model of the system that connects parameter vector errors with the tracking error.

3.2.3. Boundedness and convergence of ε

Theorem 2. *The adaptive law described by* (20), (23), *and* $m^2 = 1 + n_s^2$ *together with error model* (47), *guarantees*:

- $\varepsilon, \tilde{\mathbf{f}}, \tilde{\mathbf{q}} \in \mathscr{L}_\infty$,
- $\varepsilon, \varepsilon n_s, \varepsilon m \in \mathscr{S}(\frac{\eta^2}{m^2} + v_0^2)$, *and*
- $\dot{\mathbf{f}}, \dot{\mathbf{q}} \in \mathscr{S}(\frac{\eta^2}{m^2} + v_0^2)$

if $\eta/m \in \mathscr{L}_\infty$.

The proof is given in Appendix C.

3.2.4. Boundedness of all signals in the system and the convergence of the tracking error

How to design normalising variable m remains to be solved. Theorems 1 and 2 demand that $w/m, y_p/m, \eta/m \in \mathscr{L}_\infty$. According to (48), which defines η, we can propose the following formula

$$m^2 = 1 + n_s^2,$$

$$n_s^2 = w^2 + y_p^2 + m_s,$$

$$\dot{m}_s = -\delta_0 m_s + u^2 + y_p^2 \quad m_s(0) = 0, \quad (49)$$

where $\delta_0 > 0$ and will be discussed later.

Theorem 3. *The model reference adaptive control system, described by* (18), (20), (23) *and* (49), *is globally stable, i.e. all the signals in the system are bounded and the tracking error has the following properties*:

- $e \in \mathscr{L}_\infty$, *and*
- $e \in \mathscr{S}(\Delta_2^2 + \bar{d}^2 + v_0^2)$

if the following conditions are satisfied:

- $\frac{c}{\alpha_0^2}\Delta_\infty^2 + \frac{c}{\alpha_0^2} + c\Delta_\infty^2 < 1$,
- $c\Delta_2^2 + cv_0^2 < \delta_0$,
- $\Delta_u(s)$, $\Delta_y(s)$ *and* $G_m(s)$ *are analytic in* $\mathrm{Re}[s] \geqslant -\frac{\delta_0}{2}$,
- *reference signal* w *is continuous, and*
- $\boldsymbol{\beta}$ *is a function of continuous signals*

where

- $\Delta_\infty = \max(\|\Delta_u(s)\|_{\infty\delta_0}, \|\Delta_y(s)\|_{\infty\delta_0} + \bar{f}_y)$,
- $\Delta_2 = \max(\|\Delta_u(s)\|_{2\delta_0}, \|\Delta_y(s)\|_{2\delta_0}, \bar{f}_w, \bar{f}_y)$,
- $\bar{d} = \sup_t |d(t)|$,
- α_0 *is an arbitrary constant such that* $\alpha_0 > a_m$,
- c *are constants that depend on different system parameters (reference model,* δ_0, v_0, *and other).*

Furthermore, estimated control gains converge to the residual set:

$$\left\{ f_i, q_i \,\middle|\, |f_i| < \frac{1}{v_0},\ |q_i| < \frac{1}{v_0},\ i = 1, \dots, k \right\}. \tag{50}$$

The proof is given in Appendix C.

Remark 1. In the theorem, transfer functions in the Laplace domain are used instead of the equivalent operators in the time domain. If the analyticity or norms of the operators are needed, the description in the s domain is more suitable. If the input–output relations of the system are used, the description in the time domain is usually used. Both notations are used interchangeably in the rest of the paper.

Remark 2. If the unmodelled dynamics of the plant that are represented by Δ_∞ are small enough, then by choosing large enough α_0 (which is not a design parameter but is only used in stability proof, meaning it is arbitrary), the first condition can always be satisfied. The first term in the second condition also gives information about the unmodelled dynamics. Together with the choice of the leakage parameter v_0, the second condition represents the lower bound on δ_0 that would still ensure stable behaviour. On the other hand, δ_0 is also limited from above with the third condition. The dominant limitation of the third condition is usually the condition on $G_m(s)$ since it is not advisable to choose a reference model that is 'quicker' than parasitic dynamics due to robustness issues. The continuity of the reference signal is not as stringent as it appears at first sight. We can see from (C.44) that by choosing large enough α_0, arbitrarily large derivatives of the reference signal are allowed. Since adaptive control is usually realised by means of a digital controller, a reference signal that consists of square impulses can be treated as continuous with large derivatives at points of discontinuity.

Remark 3. The parameters f_i and q_i will converge to the residual set (50) if the adaptive error ε and the fulfilment of the corresponding membership functions β_i are non-zero. If these conditions are not satisfied, the parameters will be frozen. This means that the asymptotic convergence of the parameters is not guaranteed. On the other hand, the parameters that may be outside bounds (50) do not contribute to the control signal since β_i is explicitly present in the control law.

Remark 4. The consequence of the second property of Theorem 3 is that short bursts of the signals are possible (they are quite usual in many forms of adaptive systems, see e.g. [1]) but they are of finite amplitude and their duration is relatively short.

Remark 5. The convergence set (50) also represents potential danger in the event that the plant itself is unstable and no element of set (50) provides stable behaviour of the plant. But this is a known problem of the adaptation with leakage as shown by Rey et al. [14]. To avoid it, a controller parameterisation $(\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_k, \hat{q}_1, \hat{q}_2, \ldots, \hat{q}_k)$ has to be known that ensures stability of the system. A slightly modified adaptive law (20) should therefore be used to obtain stable system:

$$\dot{f}_i = -\gamma_{fi} b_{\text{sign}} \varepsilon w \beta_i - \gamma_{fi} |\varepsilon m| v_0 (f_i - \hat{f}_i) \beta_i, \quad i = 1, 2, \ldots, k,$$

$$\dot{q}_i = \gamma_{qi} b_{\text{sign}} \varepsilon y_p \beta_i - \gamma_{qi} |\varepsilon m| v_0 (q_i - \hat{q}_i) \beta_i, \quad i = 1, 2, \ldots, k. \tag{51}$$

Remark 6. The problem of choosing the design parameters $\mathbf{\Gamma}$, v_0 and δ_0 is still quite open. This is an eternal problem in adaptive control. Some guidelines on choosing v_0 and δ_0 can be obtained from the conditions of Theorem 3. The latter does not impose any limitations on adaptive gain, but it is generally known that its choice is of crucial importance for the good performance of the adaptive system. As always, it turns out that any prior knowledge that is available to the designer can be used to improve the performance or robustness of the overall system.

4. Simulation example

The proposed algorithm was tested on a simulation example. The simulated test plant consisted of three water tanks. The schematic representation of the plant is given in Fig. 1. The control objective was to maintain the water level in the third tank by changing the inflow into the first tank.

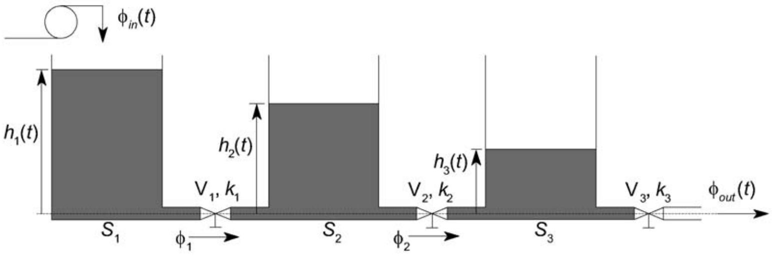

Fig. 1. Schematic representation of the plant.

When modelling the plant, it was assumed that the flow through the valve was proportional to the square root of the pressure difference on the valve. The mass conservation equations for the three tanks are:

$$
\begin{aligned}
S_1\dot{h}_1 &= \phi_{in} - k_1\,\mathrm{sgn}(h_1 - h_2)\sqrt{|h_1 - h_2|},\\
S_2\dot{h}_2 &= k_1\,\mathrm{sgn}(h_1 - h_2)\sqrt{|h_1 - h_2|} - k_2\,\mathrm{sgn}(h_2 - h_3)\sqrt{|h_2 - h_3|},\\
S_3\dot{h}_3 &= k_2\,\mathrm{sgn}(h_2 - h_3)\sqrt{|h_2 - h_3|} - k_3\,\mathrm{sgn}(h_3)\sqrt{|h_3|},
\end{aligned}
\qquad (52)
$$

where ϕ_{in} is the volume inflow into the first tank, h_1, h_2, and h_3 are the water levels in three tanks, S_1, S_2, and S_3 are areas of the tanks cross-sections, and k_1, k_2, and k_3 are coefficients of the valves. The following values were chosen for the parameters of the system:

$$
\begin{aligned}
&S_1 = S_2 = S_3 = 2 \times 10^{-2}\,\mathrm{m}^2,\\
&k_1 = k_2 = k_3 = 2 \times 10^{-4}\,\mathrm{m}^{5/2}\,\mathrm{s}^{-1}.
\end{aligned}
\qquad (53)
$$

The nominal value of inflow ϕ_{in} was set to $8 \times 10^{-5}\,\mathrm{m}^3\,\mathrm{s}^{-1}$, resulting in steady-state values 0.48, 0.32 and 0.16 m for h_1, h_2, and h_3, respectively. In the following, u and y_p denote deviations of ϕ_{in} and h_3, respectively, from the operating point.

The proposed fuzzy model reference adaptive control algorithm was compared to the classical MRAC with e_1-modification via two experiments. Adaptive gains γ_{fi} and γ_{qi} in the case of DFMRAC were the same as γ_f and γ_q, respectively, in the case of MRAC. The e_1-modification constants v_0 were also the same in both cases. A reference signal was chosen as a periodical piece-wise constant function which covered quite a wide area around the operating point ($\pm 50\%$ of the nominal value). There were 11 triangular fuzzy membership functions (the fuzzification variable was y_p) used; these were distributed evenly across the interval $[-0.1, 0.1]$. If any information regarding nonlinearity is available, it can be used in choosing the membership functions to obtain better results. The control input signal u was saturated at the interval $[-8 \times 10^{-5}, 8 \times 10^{-5}]$. No prior knowledge of the estimated parameters was available to us, so the initial parameter estimates were 0 for all examples.

The first simulation experiment assumed that the tanks were high enough so that they would never fill up. Figs. 2 and 3 show the results of the classical MRAC, while Figs. 4 and 5 show the results of DFMRAC. By comparing the responses in Figs. 2 and 4, one can observe that every change in the reference signal results in a sudden increase in tracking error e (up to 0.01). This is due to the fact that zero tracking of the reference model with relative degree 1 is not possible if the plant has relative degree 3. Otherwise, much better results are achieved when using DFMRAC since the differences in system dynamics when changing the operating point almost do not influence the responses of the system. This is clearly seen by comparing Figs. 2 and 4. Also, the oscillations in parameter estimates are smaller in the case of the fuzzy adaptive law. On the other hand, a much longer period is needed for the estimates to reach quasi-equilibrium if fuzzy adaptive law is used compared to the time needed if classical adaptive law is used.

The second experiment was conducted on the model where the tanks were 0.6 m high. The responses are shown in Figs. 6 and 7. When the water level in a tank reaches 0.6 m, the security mechanism stops the water inflow and prevents spillage. This assumption introduced discontinuity into the system. A consequence was that the meeting of control requirements was not possible. It is

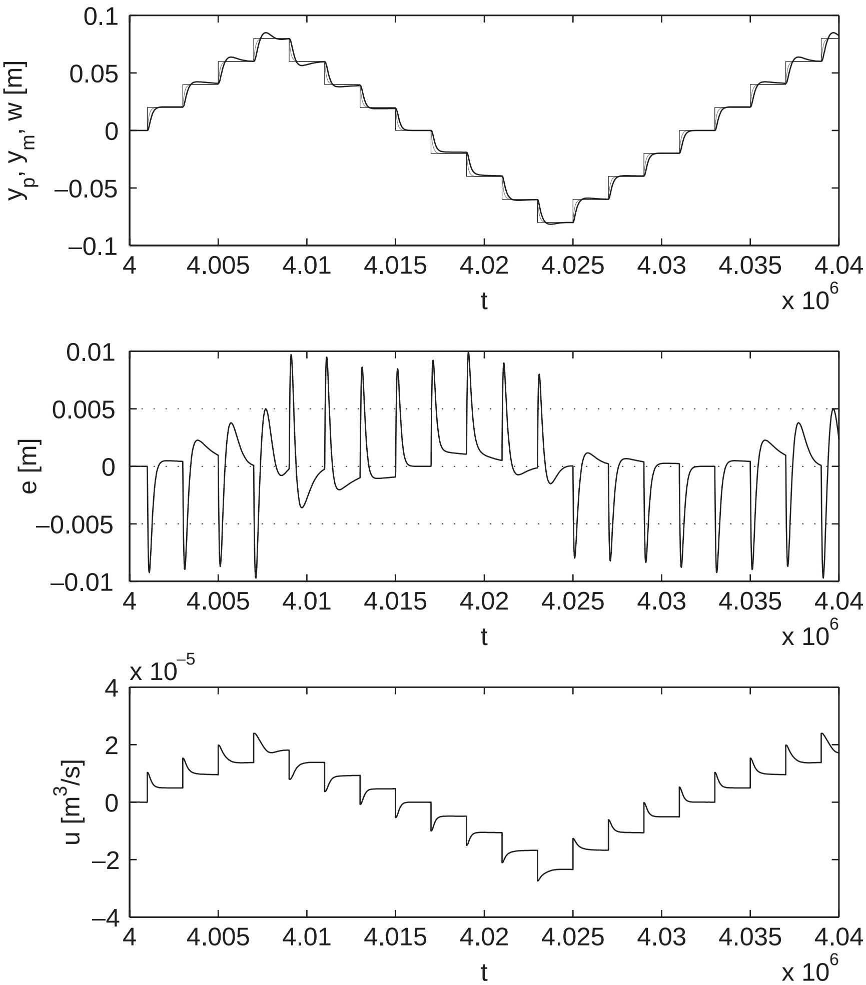

Fig. 2. The classical MRAC with e_1-modification–time plots of the reference signal and outputs of the plant and the reference model (upper figure), time plot of tracking error (middle figure), and time plot of the control signal (lower figure).

true that the water level never has to approach 0.6 m in the third tank, but it does in the first tank when the desired level in the third tank reaches some point. No control algorithm exists that could zero the tracking error when the reference signal is too high. The classical adaptive law responded to that disturbance by increasing the control parameters, while fuzzy adaptive law increased only one parameter. When the system left that operating point, the behaviour of the DFMRAC system was good, while classical MRAC had to retune the parameters to reach the normal values.

The experiments show that the performance of the DFMRAC is much better than the performance of the other approach. Very good results are obtained in the case of DFMRAC, even though that the parasitic dynamics are nonlinear and the plant of 'relative degree' 3 is forced to follow the reference model of relative degree 1.

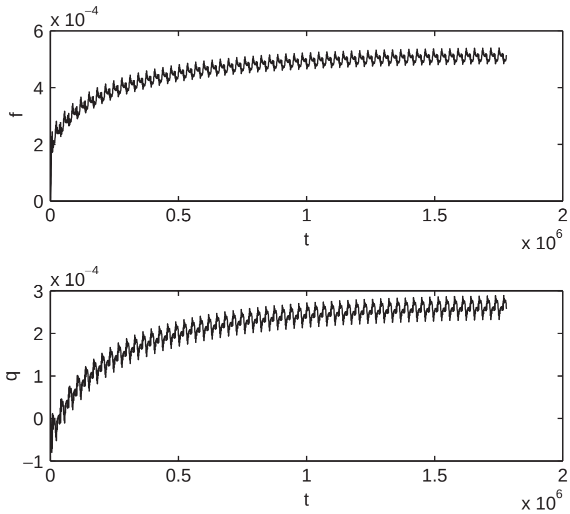

Fig. 3. The classical MRAC with e_1-modification–time plots of feedforward (upper figure) and feedback (lower figure) control gains.

The drawback of DFMRAC is relatively slow convergence, since the parameters are only adapted when the corresponding membership is non-zero. This drawback can be overcome by using classical MRAC in the beginning when there are no parameter estimates or the estimates are bad. When the system approaches reasonably good behaviour, adaptation can switch to that proposed by initialising all elements of vectors **f** and **q** by estimated scalar parameters f and q, respectively. After the switching, the fine tuning of parameters in vectors **f** and **q** is performed to accommodate the control requirements. In our case all the estimates were 0 at the beginning, resulting in the fact that the controller gains were too small at the beginning. The output of the plant was also too small and some periods of the reference signal were needed so that the output reached the outermost membership functions. This means that for some time the corresponding control gains were 0, since they were not adapted. The system was in the 'magic circle' that prevented it from reaching the desired behaviour at the beginning. Some experiments have shown that if the reference model output was chosen as a fuzzification variable, this start-up interval was shortened, which is understandable since adaptation started in all fuzzy domains in the first period. When the system was moving towards desired behaviour, the difference between y_p and y_m as fuzzification variables did not make much difference. The problem is that the approach with y_m as fuzzification variable does not have any background in the fuzzy model.

5. Conclusions

A direct fuzzy adaptive control algorithm was presented in the paper. It was shown in Theorem 3 that the closed-loop system is stable provided some conditions concerning the size of disturbances and high-order parasitics are met. The advantage of the proposed approach is that it is very simple to design, but it still offers the advantages of nonlinear and adaptive controllers. It was shown in the

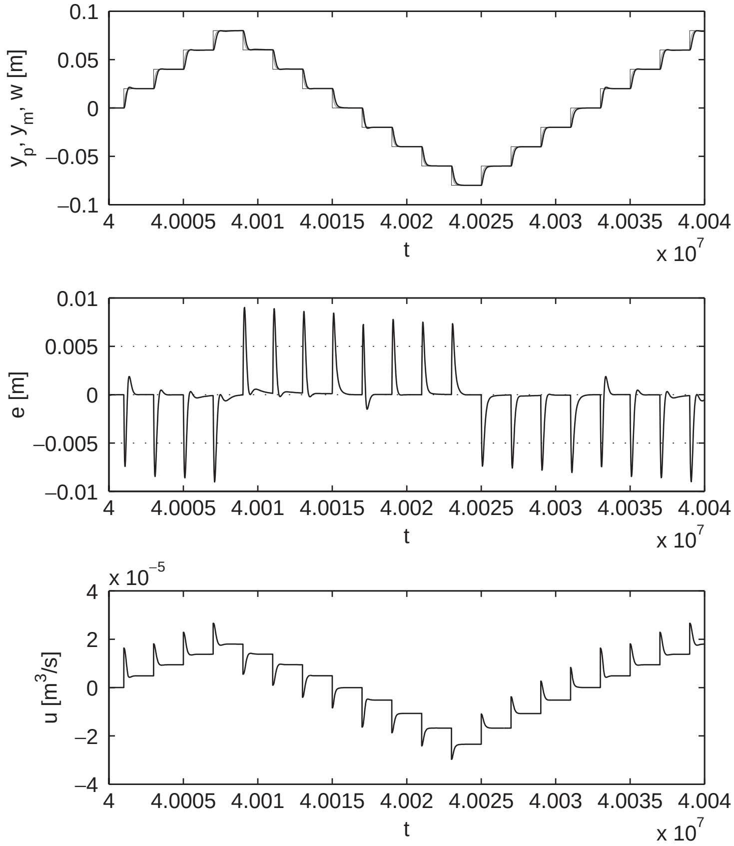

Fig. 4. The DFMRAC–time plots of the reference signal and outputs of the plant and the reference model (upper figure), time plot of tracking error (middle figure), and time plot of the control signal (lower figure).

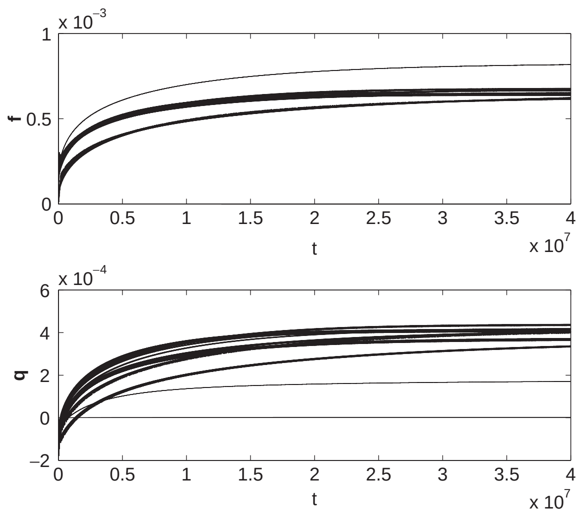

Fig. 5. The DFMRAC–time plots of feedforward (upper figure) and feedback (lower figure) control gains.

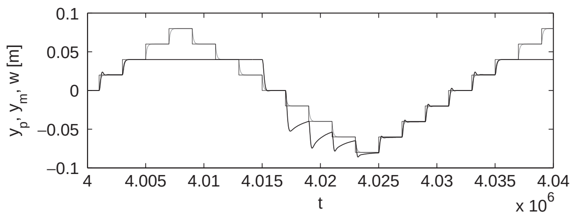

Fig. 6. Response of the classical MRAC with e_1-modification (the case with finite height of tanks).

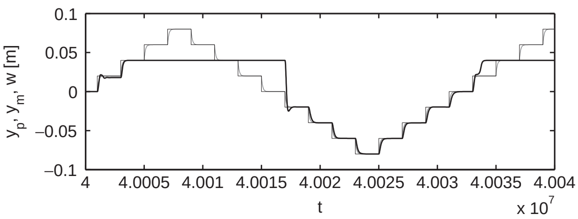

Fig. 7. Response of the DFMRAC (the case with finite height of tanks).

example that good results can be obtained if a third-order plant is treated as a first-order plant. It also proves very successful when disturbances are present only in certain operating regions, since only estimates of the corresponding parameters are bad. When the system leaves those conditions (fuzzy domains), perfect functioning of the controller is restored instantly. The drawback of the approach is the long period of adaptation, which is the result of the large number of parameters that have to be estimated. To speed up adaptation, classical adaptation can be used in the early phase, followed by fuzzy adaptation when classical adaptation quasi-settles. Switching from the former to the latter is very easy and does not cause any bumps.

Appendix A

In this section some functional analysis preliminaries are given on the norms and smallness of signals in the mean square sense that are used frequently throughout the paper. They are given here for the sake of completeness. More complete treatment can be found in textbooks on functional analysis. A very good summary needed for use in control in general, and especially in robust adaptive control, is given in the book by Ioannou and Sun [7].

Since most of the signals analysed in the paper do not have finite $\mathscr{L}_p$ norms, $\mathscr{L}_{pe}$ norms are used instead. They are defined as usual $\mathscr{L}_p$ norms, but the upper limit of the integral is t instead of infinity. If a function has a finite $\mathscr{L}_{pe}$ norm, we say it belongs to $\mathscr{L}_{pe}$ set. For stability analysis of the proposed algorithm, exponentially weighted $\mathscr{L}_2$ norms are shown to be particularly useful. They are defined as

$$\|\mathbf{x}_t\|_{2\delta} \triangleq \left(\int_0^t \mathrm{e}^{-\delta(t-\tau)}\mathbf{x}^{\mathrm{T}}(\tau)\mathbf{x}(\tau)\,\mathrm{d}\tau\right)^{1/2}, \tag{A.1}$$

where $\delta\geqslant 0$ is a constant. If the LTI system is given by

$$y = H(s)u, \tag{A.2}$$

where $H(s)$ is a proper rational function of s that is analytic in $\mathrm{Re}[s]\geqslant -\delta/2$ for some $\delta\geqslant 0$ and $u\in\mathscr{L}_{2e}$ then

$$\|y_t\|_{2\delta} \leqslant \|H(s)\|_{\infty\delta}\|u_t\|_{2\delta}, \tag{A.3}$$

where

$$\|H(s)\|_{\infty\delta} \triangleq \sup_{\omega} |H(\mathrm{j}\omega - \tfrac{\delta}{2})|. \tag{A.4}$$

Furthermore, when $H(s)$ is strictly proper, we have

$$|y(t)| \leqslant \|H(s)\|_{2\delta}\|u_t\|_{2\delta} \tag{A.5}$$

where

$$\|H(s)\|_{2\delta} \triangleq \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{\infty} |H(\mathrm{j}\omega - \tfrac{\delta}{2})|^2\,\mathrm{d}\omega\right)^{1/2}. \tag{A.6}$$

Definition of smallness in the mean square sense [7]:

Let $\mathbf{x}: [0,\infty)\to\mathbb{R}^n$, $w: [0,\infty)\to\mathbb{R}^+$ where $\mathbf{x}\in\mathscr{L}_{2e}$, $w\in\mathscr{L}_{1e}$ and consider the set

$$\mathscr{S}(w) = \left\{\mathbf{x}, w \,\middle|\, \int_t^{t+T} \mathbf{x}^{\mathrm{T}}(\tau)\mathbf{x}(\tau)\,\mathrm{d}\tau \leqslant c_0\int_t^{t+T} w(\tau)\,\mathrm{d}\tau + c_1,\ \forall t, T \geqslant 0\right\}, \tag{A.7}$$

where $c_0, c_1\geqslant 0$ are some finite constants. We say that $\mathbf{x}$ is w-small in the mean square sense if $\mathbf{x}\in\mathscr{S}(w)$.

Appendix B

Some useful lemmas are given in the appendix. They are indispensable in the paper since they are used repeatedly in the proofs of the theorems. They are given here without explicit proofs. Most of them are proven implicitly; the others are very simple to prove and therefore the proofs are omitted.

If $\mathbf{x}(t)$ is a vector and $\boldsymbol{\alpha}(t)$ is a scalar, or vice versa, then

$$\begin{aligned}\|(\mathbf{x}\boldsymbol{\alpha})_t\|_{2\delta} &= \left(\int_0^t \mathrm{e}^{-\delta(t-\tau)}\mathbf{x}^{\mathrm{T}}(\tau)\mathbf{x}(\tau)\boldsymbol{\alpha}^{\mathrm{T}}(\tau)\boldsymbol{\alpha}(\tau)\,\mathrm{d}\tau\right)^{1/2}\\ &\leqslant \left(\int_0^t \mathrm{e}^{-\delta(t-\tau)}\mathbf{x}^{\mathrm{T}}(\tau)\mathbf{x}(\tau)\,\mathrm{d}\tau\right)^{1/2}\sup_t(\boldsymbol{\alpha}^{\mathrm{T}}(t)\boldsymbol{\alpha}(t))^{1/2} = \|\mathbf{x}_t\|_{2\delta}\sup_t|\boldsymbol{\alpha}(t)|.\end{aligned} \tag{B.1}$$

If $\mathbf{x}(t)$ and $\boldsymbol{\alpha}(t)$ are vectors then

$$\|(\mathbf{x}^{\mathrm{T}}\boldsymbol{\alpha})_t\|_{2\delta} = \left(\int_0^t \mathrm{e}^{-\delta(t-\tau)}\mathbf{x}^{\mathrm{T}}(\tau)\boldsymbol{\alpha}(\tau)\boldsymbol{\alpha}^{\mathrm{T}}(\tau)\mathbf{x}(\tau)\,\mathrm{d}\tau\right)^{1/2}$$
$$\leqslant \left(\int_0^t \mathrm{e}^{-\delta(t-\tau)}\mathbf{x}^{\mathrm{T}}(\tau)\lambda_{\max}(\boldsymbol{\alpha}(\tau)\boldsymbol{\alpha}^{\mathrm{T}}(\tau))\mathbf{x}(\tau)\,\mathrm{d}\tau\right)^{1/2}$$
$$= \left(\int_0^t \mathrm{e}^{-\delta(t-\tau)}\mathbf{x}^{\mathrm{T}}(\tau)\mathbf{x}(\tau)|\boldsymbol{\alpha}(\tau)|^2\,\mathrm{d}\tau\right)^{1/2} = \|(\mathbf{x}|\boldsymbol{\alpha}|)_t\|_{2\delta}, \tag{B.2}$$

where $\lambda_{\max}(\mathbf{A})$ denotes the largest eigenvalue of the matrix $\mathbf{A}$. The only non-zero eigenvalue of the matrix $\mathbf{c}\mathbf{c}^{\mathrm{T}}$ is $|\mathbf{c}|$ for any vector $\mathbf{c}\neq\mathbf{0}$. Combining (B.1) and (B.2), we get

$$\|(\mathbf{x}^{\mathrm{T}}\boldsymbol{\alpha})_t\|_{2\delta} \leqslant \|(\mathbf{x}|\boldsymbol{\alpha}|)_t\|_{2\delta} \leqslant \|\mathbf{x}_t\|_{2\delta}\sup_t |\boldsymbol{\alpha}(t)|. \tag{B.3}$$

If $\mathbf{x}(t)$ is a vector then the upper bound on $\|\mathbf{x}_t\|_{2\delta}$ is

$$\|\mathbf{x}\|_{2\delta} = \left(\int_0^t \mathrm{e}^{-\delta(t-\tau)}\mathbf{x}^{\mathrm{T}}(\tau)\mathbf{x}(\tau)\,\mathrm{d}\tau\right)^{1/2} \leqslant \left(\int_0^t \mathrm{e}^{-\delta(t-\tau)}\,\mathrm{d}\tau\right)^{1/2} \sup_t(\mathbf{x}^{\mathrm{T}}(t)\mathbf{x}(t))^{1/2}$$
$$= \sqrt{\frac{1-\mathrm{e}^{-\delta t}}{\delta}}\sup_t |\mathbf{x}(t)| < \frac{1}{\sqrt{\delta}}\sup_t |\mathbf{x}(t)|. \tag{B.4}$$

Since elements of vector $\boldsymbol{\beta}$ are normalised it follows:

$$\sup_t |\boldsymbol{\beta}(t)| = \sup_t \sqrt{\sum_{i=1}^k \beta_i^2(t)} \leqslant \sqrt{\sum_{i=1}^k \beta_i} = 1. \tag{B.5}$$

If $f(t)$, $g(t)$ and $h(t)$ are scalar functions of time and $\mathbf{x}_i(t)$ are vector functions of time ($i = 1,2,\ldots,k$), then

- $$\|(fh)_t\| + \|(gh)_t\| = \sqrt{\int_0^t \mathrm{e}^{-\delta(t-\tau)}(f^2(\tau)+g^2(\tau))h^2(\tau)\,\mathrm{d}\tau} = \|(h\sqrt{f^2+g^2})_t\|, \tag{B.6}$$

- $$\left\|\begin{matrix}\mathbf{x}_1(t)\\ \mathbf{x}_2(t)\\ \vdots\\ \mathbf{x}_k(t)\end{matrix}\right\| = \sqrt{\int_0^t \mathrm{e}^{-\delta(t-\tau)}(\mathbf{x}_1^{\mathrm{T}}(t)\mathbf{x}_1(t)+\cdots+\mathbf{x}_k^{\mathrm{T}}(t)\mathbf{x}_k(t))\,\mathrm{d}\tau}$$
$$= \|\mathbf{x}_1(t)\| + \cdots + \|\mathbf{x}_k(t)\|. \tag{B.7}$$

If x_i $(i=1,2,\ldots,n)$ are real numbers, then

$$\left(\sum_{i=1}^{n} x_i\right)^2 \leqslant n\sum_{i=1}^{n} x_i^2. \tag{B.8}$$

Appendix C

In this appendix the extensive proofs of Theorems 2 and 3 are given. Both follow the general lines of the similar proofs presented by Ioannou and Sun [7]. There are, of course, many peculiarities of fuzzy modelling that make our proofs quite different to the ones mentioned.

The proof of Theorem 2. According to the error model (47), the tracking error e is obtained by filtering parameter errors and the unmodelled term by a reference model G_m

$$e = G_m(p)\left(\frac{b}{b_m}\tilde{\mathbf{f}}^{\mathrm{T}}\boldsymbol{\beta}w - \frac{b}{b_m}\tilde{\mathbf{q}}^{\mathrm{T}}\boldsymbol{\beta}y_p + \eta\right). \tag{C.1}$$

By combining (23) and (C.1) we get

$$\varepsilon = G_m\left(\frac{b}{b_m}\tilde{\mathbf{f}}^{\mathrm{T}}\boldsymbol{\beta}w - \frac{b}{b_m}\tilde{\mathbf{q}}^{\mathrm{T}}\boldsymbol{\beta}y_p + \eta - \varepsilon n_s^2\right). \tag{C.2}$$

A Lyapunov function is proposed

$$V = \frac{1}{2}\tilde{\mathbf{f}}^{\mathrm{T}}\boldsymbol{\Gamma}_f^{-1}\tilde{\mathbf{f}} + \frac{1}{2}\tilde{\mathbf{q}}^{\mathrm{T}}\boldsymbol{\Gamma}_q^{-1}\tilde{\mathbf{q}} + \frac{1}{2|b|}\varepsilon^2. \tag{C.3}$$

The derivative of the Lyapunov function (C.3) is

$$\dot{V} = \tilde{\mathbf{f}}^{\mathrm{T}}\boldsymbol{\Gamma}_f^{-1}\dot{\tilde{\mathbf{f}}} + \tilde{\mathbf{q}}^{\mathrm{T}}\boldsymbol{\Gamma}_q^{-1}\dot{\tilde{\mathbf{q}}} + \frac{1}{|b|}\varepsilon\dot{\varepsilon}. \tag{C.4}$$

Since $\dot{\tilde{\mathbf{f}}} = \dot{\mathbf{f}}^*$ and $\dot{\tilde{\mathbf{q}}} = \dot{\mathbf{q}}^*$, it follows from (C.4), using (21) and (C.2)

$$\begin{aligned}
\dot{V} =\; & \tilde{\mathbf{f}}^{\mathrm{T}}\boldsymbol{\Gamma}_f^{-1}(-\boldsymbol{\Gamma}_f b_{\mathrm{sign}}\varepsilon w\boldsymbol{\beta} - \boldsymbol{\Gamma}_f|\varepsilon m|v_0\mathbf{F}\boldsymbol{\beta}) \\
& + \tilde{\mathbf{q}}^{\mathrm{T}}\boldsymbol{\Gamma}_q^{-1}(\boldsymbol{\Gamma}_q b_{\mathrm{sign}}\varepsilon y_p\boldsymbol{\beta} - \boldsymbol{\Gamma}_q|\varepsilon m|v_0\mathbf{Q}\boldsymbol{\beta}) \\
& + \frac{1}{|b|}\varepsilon\left(-a_m\varepsilon + b_m\left(\frac{b}{b_m}\tilde{\mathbf{f}}^{\mathrm{T}}\boldsymbol{\beta}w - \frac{b}{b_m}\tilde{\mathbf{q}}^{\mathrm{T}}\boldsymbol{\beta}y_p - \varepsilon n_s^2 + \eta\right)\right) \\
=\; & -\tilde{\mathbf{f}}^{\mathrm{T}}b_{\mathrm{sign}}\varepsilon w\boldsymbol{\beta} - \tilde{\mathbf{f}}^{\mathrm{T}}|\varepsilon m|v_0\mathbf{F}\boldsymbol{\beta} + \tilde{\mathbf{q}}^{\mathrm{T}}b_{\mathrm{sign}}\varepsilon y_p\boldsymbol{\beta} - \tilde{\mathbf{q}}^{\mathrm{T}}|\varepsilon m|v_0\mathbf{Q}\boldsymbol{\beta} \\
& - \frac{a_m}{|b|}\varepsilon^2 + \mathrm{sgn}(b)\tilde{\mathbf{f}}^{\mathrm{T}}\boldsymbol{\beta}w\varepsilon - \mathrm{sgn}(b)\tilde{\mathbf{q}}^{\mathrm{T}}\boldsymbol{\beta}y_p\varepsilon + \frac{b_m}{|b|}(-\varepsilon^2 n_s^2 + \varepsilon\eta) \\
=\; & -|\varepsilon m|v_0(\tilde{\mathbf{f}}^{\mathrm{T}}\mathbf{F} + \tilde{\mathbf{q}}^{\mathrm{T}}\mathbf{Q})\boldsymbol{\beta} - \frac{a_m}{|b|}\varepsilon^2 + \frac{b_m}{|b|}(-\varepsilon^2 n_s^2 + \varepsilon\eta).
\end{aligned} \tag{C.5}$$

The last equality follows from the assumption **A3** that all b_i's and b have the same sign, i.e. b_{sign}.

What can be said about $-(\tilde{\mathbf{f}}^{\mathrm{T}}\mathbf{F}+\tilde{\mathbf{q}}^{\mathrm{T}}\mathbf{Q})\boldsymbol{\beta}$?

$$
\begin{aligned}
-(\tilde{\mathbf{f}}^{\mathrm{T}}\mathbf{F}+\tilde{\mathbf{q}}^{\mathrm{T}}\mathbf{Q})\boldsymbol{\beta} &= -\sum_{i=1}^{k}(\tilde{f}_i f_i\beta_i+\tilde{q}_i q_i\beta_i) = -\sum_{i=1}^{k}(\tilde{f}_i(f_i^*+\tilde{f}_i)\beta_i+\tilde{q}_i(q_i^*+\tilde{q}_i)\beta_i)\\
&= \sum_{i=1}^{k}(-\tilde{f}_i^2\beta_i-\tilde{q}_i^2\beta_i-f_i^*\tilde{f}_i\beta_i-q_i^*\tilde{q}_i\beta_i)\\
&\leqslant \sum_{i=1}^{k}(-\tilde{f}_i^2\beta_i-\tilde{q}_i^2\beta_i+|f_i^*|\cdot|\tilde{f}_i|\beta_i+|q_i^*|\cdot|\tilde{q}_i|\beta_i)\\
&\leqslant \sum_{i=1}^{k}\Big(-\tilde{f}_i^2\beta_i-\tilde{q}_i^2\beta_i+|f_i^*|\cdot|\tilde{f}_i|\beta_i+|q_i^*|\cdot|\tilde{q}_i|\beta_i\\
&\qquad +\frac{\beta_i}{2}(|\tilde{f}_i|-|f_i^*|)^2+\frac{\beta_i}{2}(|\tilde{q}_i|-|q_i^*|)^2\Big)\\
&= \sum_{i=1}^{k}\left(-\frac{\beta_i}{2}\tilde{f}_i^2-\frac{\beta_i}{2}\tilde{q}_i^2+\frac{\beta_i}{2}f_i^{*2}+\frac{\beta_i}{2}q_i^{*2}\right)\\
&= \sum_{i=1}^{k}\left(\frac{\beta_i}{2}(-\tilde{f}_i^2-\tilde{q}_i^2+f_i^{*2}+q_i^{*2})\right)\\
&\leqslant -\min\left(\sum_{i=1}^{k}\left(\frac{\beta_i}{2}(\tilde{f}_i^2+\tilde{q}_i^2)\right)\right)+\max\left(\sum_{i=1}^{k}\left(\frac{\beta_i}{2}(f_i^{*2}+q_i^{*2})\right)\right)\\
&= 0+\max_i\left(\frac{1}{2}(f_i^{*2}+q_i^{*2})\right).
\end{aligned}
\tag{C.6}
$$

The calculated upper bound of $-(\tilde{\mathbf{f}}^{\mathrm{T}}\mathbf{F}+\tilde{\mathbf{q}}^{\mathrm{T}}\mathbf{Q})\boldsymbol{\beta}$ in (C.6) will be denoted by $\boldsymbol{\theta}^{*2}$. Using (C.6) and the inequality

$$
-\frac{a_m}{|b|}\varepsilon^2-\frac{b_m}{|b|}\varepsilon^2 n_s^2 \leqslant -\frac{\min(a_m,b_m)}{|b|}\varepsilon^2(1+n_s^2) = -\frac{\min(a_m,b_m)}{|b|}\varepsilon^2 m^2,
$$

it follows from (C.5):

$$
\begin{aligned}
\dot{V} &\leqslant |\varepsilon m|v_0\boldsymbol{\theta}^{*2}-\frac{\min(a_m,b_m)}{|b|}\varepsilon^2 m^2+\frac{b_m}{|b|}\varepsilon\eta\\
&\leqslant |\varepsilon m|\left(v_0\boldsymbol{\theta}^{*2}-\frac{\min(a_m,b_m)}{|b|}|\varepsilon m|+\frac{b_m}{|b|}\frac{|\eta|}{m}\right).
\end{aligned}
\tag{C.7}
$$

Since the desired control parameters (f_i^* and q_i^*) are finite, so is the constant $\boldsymbol{\theta}^{*2}$. The last term in the inequality (C.7) is bounded by the assumption of the theorem. The derivative of the Lyapunov function will be definitely non-positive if

$$|\varepsilon m| > \frac{v_0|b|}{\min(a_m,b_m)}\boldsymbol{\theta}^{*2} + \frac{b_m}{\min(a_m,b_m)}\frac{|\eta|}{m}. \tag{C.8}$$

Since $|\varepsilon m|$ is positive if inequality (C.8) holds, $\dot{V}$ in (C.7) is strictly negative, not just non-positive when condition (C.8) is satisfied. Since $m \geqslant 1$ by construction, $|\varepsilon| \leqslant |\varepsilon m|$ is true and large enough $|\varepsilon|$ causes the Lyapunov function to start decreasing. It was shown previously (see Theorem 1) that $\tilde{\mathbf{f}}$ and $\tilde{\mathbf{q}}$ are bounded. From these two facts it follows:

$$V, \varepsilon, \tilde{\mathbf{f}}, \tilde{\mathbf{q}} \in \mathscr{L}_\infty. \tag{C.9}$$

Inequality (C.7) can be rewritten as

$$\begin{aligned}
\dot{V} &\leqslant -k_1^2\varepsilon^2 m^2 + v_0|\varepsilon m|\boldsymbol{\theta}^{*2} + k_2^2|\varepsilon m|\frac{|\eta|}{m} \\
&\leqslant -k_1^2\varepsilon^2 m^2 + v_0|\varepsilon m|\boldsymbol{\theta}^{*2} + k_2^2|\varepsilon m|\frac{|\eta|}{m} + \frac{1}{2}\left(k_1|\varepsilon m| - \frac{1}{k_1}\left(k_2^2\frac{|\eta|}{m} + v_0\boldsymbol{\theta}^{*2}\right)\right)^2 \\
&= -\frac{k_1^2}{2}\varepsilon^2 m^2 + \frac{1}{2k_1^2}\left(k_2^2\frac{|\eta|}{m} + v_0\boldsymbol{\theta}^{*2}\right)^2 \leqslant -\frac{k_1^2}{2}\varepsilon^2 m^2 + \frac{1}{2k_1^2}\left(k_2^2\frac{|\eta|}{m} + v_0\boldsymbol{\theta}^{*2}\right)^2 \\
&\quad + \frac{1}{2k_1^2}\left(k_2^2\frac{|\eta|}{m} - v_0\boldsymbol{\theta}^{*2}\right)^2 = -\frac{k_1^2}{2}\varepsilon^2 m^2 + \frac{k_2^4}{k_1^2}\frac{|\eta|^2}{m^2} + \frac{1}{k_1^2}v_0^2\boldsymbol{\theta}^{*4},
\end{aligned} \tag{C.10}$$

where $\min(a_m,b_m)/|b|$ was substituted by k_1^2 and $b_m/|b|$ by k_2^2. Integrating both sides of the inequality (C.10), we obtain

$$\int_{t_0}^{t}\frac{k_1^2}{2}\varepsilon^2 m^2\,\mathrm{d}\tau \leqslant \int_{t_0}^{t}\left(\frac{k_2^4}{k_1^2}\frac{\eta^2}{m^2} + \frac{1}{k_1^2}v_0^2\boldsymbol{\theta}^{*4}\right)\mathrm{d}\tau + V(t_0) - V(t) \tag{C.11}$$

for $\forall t \geqslant t_0$ and any $t_0 \geqslant 0$. Because $V \in \mathscr{L}_\infty$ and $m^2 = 1 + n_s^2$, it follows

$$\varepsilon, \varepsilon n_s, \varepsilon m \in \mathscr{S}\left(\frac{\eta^2}{m^2} + v_0^2\right), \tag{C.12}$$

where $\mathscr{S}(\cdot)$ gives information about the mean square value of the signals and is defined in Appendix A.

From (21) it follows:

$$\begin{aligned}
\dot{\mathbf{f}} &= -\boldsymbol{\Gamma}_f\varepsilon m\left(b_{\text{sign}}\frac{w}{m} + \operatorname{sgn}(\varepsilon m)v_0\mathbf{F}\right)\boldsymbol{\beta}, \\
\dot{\mathbf{q}} &= -\boldsymbol{\Gamma}_q\varepsilon m\left(-b_{\text{sign}}\frac{y_p}{m} + \operatorname{sgn}(\varepsilon m)v_0\mathbf{Q}\right)\boldsymbol{\beta}
\end{aligned} \tag{C.13}$$

and consecutively

$$|\dot{\mathbf{f}}| \leqslant c|\varepsilon m| \quad \text{since } \frac{w}{m},\ \mathbf{F} \in \mathscr{L}_\infty,$$
$$|\dot{\mathbf{q}}| \leqslant c|\varepsilon m| \quad \text{since } \frac{y_p}{m},\ \mathbf{Q} \in \mathscr{L}_\infty. \tag{C.14}$$

Combining (C.12) and (C.14) it follows $\dot{\mathbf{f}}, \dot{\mathbf{q}} \in \mathscr{S}(\frac{\eta^2}{m^2} + v_0^2)$. □

The proof of Theorem 3. In the following, $\|(\cdot)\|$ denotes the $\mathscr{L}_{2\delta_0}$ norm, i.e. $\|(\cdot)_t\|_{2\delta_0}$.

By defining $\tilde{\boldsymbol{\theta}}^{\mathrm{T}} \triangleq [\tilde{\mathbf{f}}^{\mathrm{T}}\ \tilde{\mathbf{q}}^{\mathrm{T}}]$ and $\boldsymbol{\omega}^{\mathrm{T}} \triangleq [\boldsymbol{\beta}^{\mathrm{T}} w\ -\boldsymbol{\beta}^{\mathrm{T}} y_p]$, (47) can be rewritten as

$$e = G_m(p)\left(\frac{b}{b_m}\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\omega} + \eta\right). \tag{C.15}$$

The normalising signal m in (49) is equal to

$$m^2 = 1 + w^2 + y_p^2 + \|u\|^2 + \|y_p\|^2. \tag{C.16}$$

It will be shown that $\eta/m, \|\eta\|/m, u/m, \|u\|/m, y_p/m, \|y_p\|/m, \omega/m, \|\omega\|/m, \|\dot{y}_p\|/m \in \mathscr{L}_\infty$. If additionally $\dot{w} \in \mathscr{L}_\infty$, then $\frac{\|\dot{\boldsymbol{\omega}}\|}{m} \in \mathscr{L}_\infty$.

It follows from (48), by using property (A.3):

$$\begin{aligned}\|\eta\| &\leqslant \|\Delta_u\|_{\infty\delta_0}\|u\| + \|\Delta_y\|_{\infty\delta_0}\|y_p\| + \bar{f}_w\|w\| + \bar{f}_y\|y_p\| + \|d\| \\ &\leqslant \frac{1}{\sqrt{\delta}}(\bar{f}_w\bar{w} + \bar{d}) + \|\Delta_u\|_{\infty\delta_0}\|u\| + (\|\Delta_y\|_{\infty\delta_0} + \bar{f}_y)\|y_p\| \\ &\leqslant \frac{1}{\sqrt{\delta}}(\bar{f}_w\bar{w} + \bar{d}) + \Delta_\infty m,\end{aligned} \tag{C.17}$$

where $\Delta_\infty = \max(\|\Delta_u\|_{\infty\delta_0}, \|\Delta_y\|_{\infty\delta_0} + \bar{f}_y)$. Similarly it follows from (48), (A.5) and (B.8):

$$\begin{aligned}|\eta| &\leqslant \|\Delta_u(s)\|_{2\delta_0}\|u_t\|_{2\delta_0} + \|\Delta_y(s)\|_{2\delta_0}\|(y_p)_t\|_{2\delta_0} + \bar{f}_w|w| + \bar{f}_y|y_p| + \bar{d} \\ &\leqslant \max(\|\Delta_u(s)\|_{2\delta_0}, \|\Delta_y(s)\|_{2\delta_0}, \bar{f}_w, \bar{f}_y)(\|u_t\|_{2\delta_0} + \|(y_p)_t\|_{2\delta_0} + |w| + |y_p|) + \bar{d} \\ &\leqslant 2\max(\|\Delta_u(s)\|_{2\delta_0}, \|\Delta_y(s)\|_{2\delta_0}, \bar{f}_w, \bar{f}_y)m + \bar{d} = 2\Delta_2 m + \bar{d},\end{aligned} \tag{C.18}$$

where $\Delta_2 = \max(\|\Delta_u(s)\|_{2\delta_0}, \|\Delta_y(s)\|_{2\delta_0}, \bar{f}_w, \bar{f}_y)$. From (C.18) and (B.8) we have

$$\eta^2 \leqslant 8\Delta_2^2 m^2 + 2\bar{d}^2. \tag{C.19}$$

The boundedness of $\|u\|/m$, $\|y_p\|/m$ and y_p/m follows directly from (C.16). Using (B.7), (B.1), (B.5), (B.4) and (C.16), we get

$$\|\boldsymbol{\omega}\| = \|-\boldsymbol{\beta}w\| + \|-\boldsymbol{\beta}y_p\| \leqslant \|w\| + \|y_p\| \leqslant c\bar{w} + m. \tag{C.20}$$

Similarly:

$$|\boldsymbol{\omega}| = \sqrt{|\boldsymbol{\beta} w|^2 + |-\boldsymbol{\beta} y_p|^2} = \sqrt{|\boldsymbol{\beta}|^2 w^2 + |\boldsymbol{\beta}|^2 y_p^2} \leqslant \sqrt{w^2 + y_p^2} < m. \tag{C.21}$$

The input signal u is calculated according to the formula

$$u = \boldsymbol{\theta}^{\mathrm{T}} \boldsymbol{\omega}. \tag{C.22}$$

Due to the boundedness of the parameter vector $\boldsymbol{\theta}$ that is guaranteed by the adaptive law (Theorem 1) and (C.21), it can be concluded that $u \leqslant cm$.

Using (C.15), the output of the plant can be written as

$$y_p = G_m(p)\left(w + \frac{b}{b_m}\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\omega} + \eta\right). \tag{C.23}$$

The consequence of (C.23) is

$$\dot{y}_p = pG_m(p)\left(w + \frac{b}{b_m}\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\omega} + \eta\right). \tag{C.24}$$

From (C.24) it follows, by using (A.3), (B.4) and (C.17)

$$\begin{aligned} \|\dot{y}_p\| &\leqslant \|sG_m(s)\|_{\infty\delta_0}\left(\|w\| + \frac{b}{b_m}\|\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\omega}\| + \|\eta\|\right) \leqslant c\bar{w} + c\|\boldsymbol{\omega}\| + c\|\eta\| \\ &\leqslant c\bar{w} + c\bar{f}_w\bar{w} + c\bar{d} + c\Delta_\infty m + cm. \end{aligned} \tag{C.25}$$

The upper bound for the norm of the vector $\dot{\boldsymbol{\omega}}$ is calculated from the norms of its elements (see Eq. (B.7))

$$\begin{aligned} \|\dot{\boldsymbol{\omega}}\| &= \left\| \begin{matrix} \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\beta} w) \\ \frac{\mathrm{d}}{\mathrm{d}t}(-\boldsymbol{\beta} y_p) \end{matrix} \right\| = \|\dot{\boldsymbol{\beta}} w + \boldsymbol{\beta}\dot{w}\| + \|-\boldsymbol{\beta}\dot{y}_p - \dot{\boldsymbol{\beta}} y_p\| \\ &\leqslant \|\dot{\boldsymbol{\beta}} w\| + \|\boldsymbol{\beta}\dot{w}\| + \|\boldsymbol{\beta}\dot{y}_p\| + \|\dot{\boldsymbol{\beta}} y_p\| \\ &\leqslant \sup_t |\dot{\boldsymbol{\beta}}|\left(\frac{\bar{w}}{\sqrt{\delta}} + \|y_p\|\right) + \sup_t |\boldsymbol{\beta}|\left(\frac{\bar{\dot{w}}}{\sqrt{\delta}} + \|\dot{y}_p\|\right) \\ &\leqslant c\bar{w} + c\bar{f}_w\bar{w} + c\bar{\dot{w}} + c\bar{d} + c\Delta_\infty m + cm, \end{aligned} \tag{C.26}$$

where $\bar{\dot{w}} = \sup_t |\dot{w}(t)|$, i.e. reference signal $w(t)$ has to be continuous. When the membership functions depend only on signals that are continuous (e.g. y_p and w when the above assumption holds), β_i, $i = 1, 2, \ldots, k$, are also continuous and their derivatives are finite all the time, so the last inequality in (C.26) finally follows.

It follows from (C.23) and (C.17)

$$\begin{aligned} \|y_p\| &= \|G_m(s)\|_{\infty\delta_0}\left(\|w\| + \frac{b}{b_m}\|\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\omega}\| + \|\eta\|\right) \\ &\leqslant c\bar{w} + c\|\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\omega}\| + c\bar{f}_w\bar{w} + c\bar{d} + c\Delta_\infty m \end{aligned} \tag{C.27}$$

and

$$|y_p| = \|G_m(s)\|_{2\delta_0}\left(\|w\| + \frac{b}{b_m}\|\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\omega}\| + \|\eta\|\right) \leqslant c\bar{w} + c\|\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\omega}\| + c\bar{f}_w\bar{w} + c\bar{d} + c\Delta_\infty m. \quad \text{(C.28)}$$

From (11) it follows:

$$\begin{aligned} u &= \frac{1}{\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b}}(\dot{y}_p + (\boldsymbol{\beta}^{\mathrm{T}}\mathbf{a})y_p + \Delta'_y(p)y_p - \Delta'_u(p)u - d') \\ &= \left(\frac{1}{\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b}}\, pG_m(p) + \frac{\boldsymbol{\beta}^{\mathrm{T}}\mathbf{a}}{\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b}}\, G_m(p)\right)\left(w + \frac{b}{b_m}\,\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\omega} + \eta\right) \\ &\quad + \frac{b_m}{\boldsymbol{\beta}^{\mathrm{T}}\mathbf{b}}(\Delta_y(p)y_p - \Delta_u(p)u - d) \end{aligned} \quad \text{(C.29)}$$

and further:

$$\begin{aligned} \|u\| &\leqslant \left(\sup_t\left|\frac{1}{\boldsymbol{\beta}^{\mathrm{T}}(t)\mathbf{b}}\right|\|sG_m(s)\|_{\infty\delta_0} + \sup_t\left|\frac{\boldsymbol{\beta}^{\mathrm{T}}(t)\mathbf{a}}{\boldsymbol{\beta}^{\mathrm{T}}(t)\mathbf{b}}\right|\|G_m(s)\|_{\infty\delta_0}\right)\left(\|w\| + \frac{b}{b_m}\|\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\omega}\| + \|\eta\|\right) \\ &\quad + \sup_t\left|\frac{b_m}{\boldsymbol{\beta}^{\mathrm{T}}(t)\mathbf{b}}\right|(\|\Delta_y(s)\|_{\infty\delta_0}\|y_p\| + \|\Delta_u(s)\|_{\infty\delta_0}\|u\| + \|d\|) \\ &\leqslant c\bar{w} + c\bar{f}_w\bar{w} + c\bar{d} + c\Delta_\infty m + c\|\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\omega}\|. \end{aligned} \quad \text{(C.30)}$$

Combining (C.27), (C.28) and (C.30), and using (B.8), the following inequality is obtained:

$$m^2 = 1 + w^2 + y_p^2 + \|u\|^2 + \|y_p\|^2 \leqslant 1 + c\bar{w}^2 + c\bar{f}_w^2\bar{w}^2 + c\bar{d}^2 + c\Delta_\infty^2 m^2 + c\|\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\omega}\|^2. \quad \text{(C.31)}$$

From (C.2), the error ε can be rewritten as

$$\varepsilon = G_m\left(\frac{b}{b_m}\,\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\omega} - \varepsilon n_s^2 + \eta\right). \quad \text{(C.32)}$$

The product $\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\omega}$ can be decomposed into

$$\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\omega} = \frac{1}{p+\alpha_0}(\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\dot{\boldsymbol{\omega}} + \dot{\tilde{\boldsymbol{\theta}}}^{\mathrm{T}}\boldsymbol{\omega}) + \frac{\alpha_0}{p+\alpha_0}\,\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\omega}, \quad \text{(C.33)}$$

where α_0 is an arbitrary positive number.

We can use (C.32) and the fact that $G_m(s) = b_m/(s+a_m)$ to further derive from (C.33):

$$\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\omega} = \frac{1}{p+\alpha_0}(\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\dot{\boldsymbol{\omega}} + \dot{\tilde{\boldsymbol{\theta}}}^{\mathrm{T}}\boldsymbol{\omega}) + \frac{\alpha_0(p+a_m)}{(p+\alpha_0)b}\,\varepsilon - \frac{\alpha_0 b_m}{(p+\alpha_0)b}\,\eta + \frac{\alpha_0 b_m}{(p+\alpha_0)b}\,\varepsilon n_s^2. \quad \text{(C.34)}$$

The δ-shifted norms H_∞ of the transfer functions $1/(s+\alpha_0)$ and $(s+a_m)/(s+\alpha_0)$ are $1/(\alpha_0-\delta/2)$ and 1, respectively. Since $\alpha_0 > a_m > \delta/2 > 0$, it follows:

$$\frac{1}{\alpha_0 - \delta/2} < \frac{c}{\alpha_0}. \quad \text{(C.35)}$$

Using this, the following inequality is obtained:

$$\|\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\omega}\| \leqslant \frac{c}{\alpha_0}(\|\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\dot{\boldsymbol{\omega}}\| + \|\dot{\tilde{\boldsymbol{\theta}}}^{\mathrm{T}}\boldsymbol{\omega}\|) + c\alpha_0\|\varepsilon\| + c\|\eta\| + c\|\varepsilon n_s^2\|. \tag{C.36}$$

Using (B.2) and (C.21), we get

$$\|\dot{\tilde{\boldsymbol{\theta}}}^{\mathrm{T}}\boldsymbol{\omega}\| \leqslant \|\dot{\tilde{\boldsymbol{\theta}}}|\boldsymbol{\omega}|\| \leqslant \|\dot{\tilde{\boldsymbol{\theta}}}m\|. \tag{C.37}$$

From (B.3) and (C.26) it follows:

$$\|\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\dot{\boldsymbol{\omega}}\| \leqslant \|\dot{\boldsymbol{\omega}}\| \sup_t |\tilde{\boldsymbol{\theta}}| \leqslant c\bar{w} + c\bar{f}_w\bar{w} + c\bar{\dot{w}} + c\bar{\dot{d}} + c\Delta_\infty m + cm. \tag{C.38}$$

By inserting (C.37), (C.38) and (C.17) into (C.36), we get

$$\begin{aligned}\|\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\omega}\| \leqslant{}& \frac{c}{\alpha_0}\|\dot{\tilde{\boldsymbol{\theta}}}m\| + c\alpha_0\|\varepsilon\| + c\|\varepsilon n_s^2\| + \frac{c}{\alpha_0}\bar{w} + \frac{c}{\alpha_0}\bar{f}_w\bar{w} + \frac{c}{\alpha_0}\bar{\dot{w}} + \frac{c}{\alpha_0}\bar{\dot{d}}\\ &+ \frac{c}{\alpha_0}\Delta_\infty m + \frac{c}{\alpha_0}m + c\bar{f}_w\bar{w} + c\bar{d} + c\Delta_\infty m.\end{aligned} \tag{C.39}$$

Since ε is bounded (which is guaranteed by the adaptive law as shown before) and $n_s < m$, it follows:

$$\begin{aligned}\|\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\omega}\| \leqslant{}& \frac{c}{\alpha_0}\|\dot{\tilde{\boldsymbol{\theta}}}m\| + c\|\varepsilon n_s m\| + \left(\frac{c}{\alpha_0}\Delta_\infty + \frac{c}{\alpha_0} + c\Delta_\infty\right)m\\ &+ \left(c\alpha_0\bar{\varepsilon} + \frac{c}{\alpha_0}\bar{w} + \frac{c}{\alpha_0}\bar{f}_w\bar{w} + \frac{c}{\alpha_0}\bar{\dot{w}} + \frac{c}{\alpha_0}\bar{d} + c\bar{f}_w\bar{w} + c\bar{d}\right).\end{aligned} \tag{C.40}$$

Using (B.6) we get

$$\frac{c}{\alpha_0}\|\dot{\tilde{\boldsymbol{\theta}}}m\| + c\|\varepsilon n_s m\| = \frac{c}{\alpha_0}\||\dot{\tilde{\boldsymbol{\theta}}}|m\| + c\|\varepsilon n_s m\| \leqslant c\|gm\|, \tag{C.41}$$

where $g^2 = |\dot{\tilde{\boldsymbol{\theta}}}|^2/\alpha_0^2 + (\varepsilon n_s)^2$. Since $\varepsilon n_s, \dot{\tilde{\boldsymbol{\theta}}} \in \mathscr{S}(\eta^2/m^2 + v_0^2)$, it also holds that $g \in \mathscr{S}(\eta^2/m^2 + v_0^2)$ or by using (C.19)

$$g \in \mathscr{S}\left(\Delta_2^2 + \frac{\bar{d}^2}{m^2} + v_0^2\right). \tag{C.42}$$

If the term in the parentheses in (C.40) is denoted by c', the inequality (C.40) becomes

$$\|\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\boldsymbol{\omega}\| \leqslant c\|gm\| + \left(\frac{c}{\alpha_0}\Delta_\infty + \frac{c}{\alpha_0} + c\Delta_\infty\right)m + c'. \tag{C.43}$$

By using (C.31) and (C.43) it follows:

$$m^2 \leqslant 1 + c\bar{w}^2 + c\bar{f}_w^2\bar{w}^2 + c\bar{d}^2 + c\Delta_\infty^2 m^2 + c\|gm\|^2 + \left(\frac{c}{\alpha_0^2}\Delta_\infty^2 + \frac{c}{\alpha_0^2} + c\Delta_\infty^2\right)m^2 + cc'^2$$

$$\leqslant c\|gm\|^2 + \left(\frac{c}{\alpha_0^2}\Delta_\infty^2 + \frac{c}{\alpha_0^2} + c\Delta_\infty^2\right)m^2 + \left(c + \frac{c}{\alpha_0^2}\right)\bar{w}^2 + \left(c + \frac{c}{\alpha_0^2}\right)\bar{f}_w^2\bar{w}^2$$

$$+ \left(c + \frac{c}{\alpha_0^2}\right)\bar{d}^2 + c\alpha_0^2\bar{\varepsilon}^2 + \frac{c}{\alpha_0^2}\bar{\dot{w}}^2 + 1. \tag{C.44}$$

If the following condition is fulfilled

$$\frac{c}{\alpha_0^2}\Delta_\infty^2 + \frac{c}{\alpha_0^2} + c\Delta_\infty^2 < 1 \tag{C.45}$$

we have

$$m^2 \leqslant c\|gm\|^2 + \left(c + \frac{c}{\alpha_0^2}\right)\bar{w}^2 + \left(c + \frac{c}{\alpha_0^2}\right)\bar{f}_w^2\bar{w}^2$$

$$+ \left(c + \frac{c}{\alpha_0^2}\right)\bar{d}^2 + c\alpha_0^2\bar{\varepsilon}^2 + \frac{c}{\alpha_0^2}\bar{\dot{w}}^2 + c. \tag{C.46}$$

Eq. (C.46) can be rewritten by using the definition of the $\mathscr{L}_{2\delta}$ norm

$$m^2(t) \leqslant c\int_0^t \mathrm{e}^{-\delta(t-\tau)}g^2(\tau)m^2(\tau)\,\mathrm{d}\tau + K, \tag{C.47}$$

where the definition of K follows directly from (C.46). By applying the Bellman–Gronwall lemma to inequality (C.47), we get

$$m^2(t) \leqslant K\mathrm{e}^{-\delta t}\mathrm{e}^{c\int_0^t g^2(s)\,\mathrm{d}s} + K\delta\int_0^t \mathrm{e}^{-\delta(t-\tau)}\mathrm{e}^{c\int_\tau^t g^2(s)\,\mathrm{d}s}\,\mathrm{d}\tau. \tag{C.48}$$

Because of (C.42), the following is true:

$$c\int_\tau^t g^2(s)\,\mathrm{d}s \leqslant c_0 + c_1(t-\tau)\Delta_2^2 + c_2(t-\tau)\frac{\bar{d}^2}{m^2} + c_3(t-\tau)v_0^2 \tag{C.49}$$

for $\forall\tau > 0$, $\forall t > \tau$ and some positive constants c_0, c_1, c_2 and c_3. If

$$c_1\Delta_2^2 + c_2\frac{\bar{d}^2}{m^2} + c_3 v_0^2 \leqslant \delta_0 \tag{C.50}$$

then it follows from (C.48) that $m(t)$ is bounded. The second term becomes arbitrarily small as soon as $y_p(t)$ (which is smaller than $m(t)$ by design) reaches some level that depends on the upper bound of the disturbance. That term can be left out and the condition (C.50) then becomes

$$c_1\Delta_2^2 + c_3 v_0^2 < \delta_0. \tag{C.51}$$

As mentioned before, $m(t)$ will be bounded if inequality (C.51) is satisfied and $m(t)$ is large enough (this is true if $y_p(t)$ is also large enough), so that $\bar{d}^2/m^2$ is negligible. When $m(t)$ falls below the critical value, the system can temporarily become unstable, but it stabilises as soon as (C.50) is fulfilled again. This is the well-known phenomenon of bursting.

Inequality (C.51) bounds the selection of proper δ_0 in the adaptive law from below. On the other hand, δ_0 should not be too large since some transfer functions have to be analytical in the part of the complex plane where $\mathrm{Re}[s] \geqslant -\delta_0/2$.

The only task that remains unsolved is to show the convergence of the tracking error. Due to (23) the tracking error equals

$$e = \varepsilon + G_m(p)(\varepsilon n_s^2). \tag{C.52}$$

The input to the reference model εn_s^2 can be written as a product of εn_s which belongs to $\mathscr{S}(\Delta_2^2 + \bar{d}^2/m^2 + v_0^2)$, and n_s, which was shown to be bounded. It can therefore be concluded:

$$\varepsilon n_s^2 \in \mathscr{S}\left(\Delta_2^2 + \frac{\bar{d}^2}{m^2} + v_0^2\right). \tag{C.53}$$

If the impulse response of the linear system $H(p)$ belongs to $\mathscr{L}_1$ then $u' \in \mathscr{S}(\mu)$ implies that $y' \in \mathscr{S}(\mu)$ and $y \in \mathscr{L}_\infty$ for any finite $\mu \geqslant 0$ where u' and y' are the input and the output of the system $H(p)$, respectively [7]. In our case, the impulse response of the reference model is $b_m \mathrm{e}^{-a_m t}$ and therefore it belongs to $\mathscr{L}_1$. Using this fact and (C.53), it follows:

$$G_m(p)(\varepsilon n_s^2) \in \mathscr{S}\left(\Delta_2^2 + \frac{\bar{d}^2}{m^2} + v_0^2\right),$$

$$G_m(p)(\varepsilon n_s^2) \in \mathscr{L}_\infty. \tag{C.54}$$

It was shown previously that

$$\varepsilon \in \mathscr{S}\left(\Delta_2^2 + \frac{\bar{d}^2}{m^2} + v_0^2\right),$$

$$\varepsilon \in \mathscr{L}_\infty. \tag{C.55}$$

By combining (C.52), (C.54) and (C.55), we arrive at the final result

$$e \in \mathscr{S}(\Delta_2^2 + \bar{d}^2 + v_0^2),$$

$$e \in \mathscr{L}_\infty, \tag{C.56}$$

where it was taken into account that m is bounded.

The proof of (50) follows directly from the proof of Theorem 1 (see Eq. (27)) by noting that $|w/m| < 1$ and $|y_p/m| < 1$. □

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